Modeling and Optimal Control of Nonlinear Underactuated Mechanical Systems – a Survey

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***Abstract*—The purpose of this paper is to examine the state-of-the-art in the analysis and control of nonlinear underactuated mechanical systems. Typical representatives of such systems are presented together with theoretical fundamentals. Application of optimal control techniques and hybrid systems theory to underactuated systems is described. Future research challenges are suggested and a list of essential references is included.**

***Keywords*—nonlinear underactuated mechanical systems, Lagrangian mechanics, optimal control techniques, hybrid systems theory**

I. Introduction

Underactuated systems represent a significant group of mechanical systems which range from simple planar robots or inverted pendulum systems to advanced higher-order systems with applications in robotics and air/sea transport [1]. In general, these systems are inherently nonlinear and have fewer control inputs than degrees of freedom [2], which presents a significant challenge to modeling and controller design [3].

This paper aims to provide a concise survey of the main achieved results and applications of underactuated systems with frequent references to crucial works in the field. After a brief summary of mathematical and physical preliminaries, principal examples of underactuated mechanical systems are presented. The ability of optimal control techniques to suit the properties of underactuated systems is next evaluated, and the potential of hybrid systems theory, which describes the integration of continuous/discrete dynamics in a dynamical system, is briefly examined with regard to modeling and control of underactuated systems.

All the way throughout the paper, open research problems are indicated. Most of these explore the theoretical and practical aspects of mutual overlaps between the physics of underactuated systems, optimal control techniques, and hybrid systems theory [1].

II. Modeling of Nonlinear Underactuated Systems using Lagrangian Mechanics

According to the Lagrangian formulation of classical mechanics, every possible configuration of a multi-body mechanical system can be described by a vector of generalized coordinates $\theta(t)$, which correspond to the degrees of freedom (DoFs) of the system. Using the d'Alembert maximum principle, *Euler-Lagrange equations* were derived (one equation of motion is specified for every DoF) [1][4]:

$$\frac{d}{dt}\left(\frac{\partial L(t)}{\partial \dot{\boldsymbol{\theta}}(t)}\right)-\frac{\partial L(t)}{\partial \boldsymbol{\theta}(t)}+\frac{\partial D(t)}{\partial \dot{\boldsymbol{\theta}}(t)}=\boldsymbol{Q}^*(t) \tag{1}$$

where $L(t)$ is the difference between multi-body system's kinetic and potential energies (each given as a sum of energies of individual bodies), $D(t)$ stands for the dissipation properties and $\boldsymbol{Q}^*(t)$ is the vector of generalized external inputs. The process of mathematical model derivation via (1) is naturally transformable into a general algorithm which can be implemented using accessible symbolic software packages such as MATLAB's *Symbolic Math Toolbox*, *Maple*, and *Wolfram Mathematica* [1]. The mathematical model of a general controllable mechanical system derived from (1) is given as a following set of second-order differential equations:

$$\ddot{\boldsymbol{\theta}}(t)=\boldsymbol{f}\left(\boldsymbol{\theta}(t),\,\dot{\boldsymbol{\theta}}(t),\,\boldsymbol{u}(t),\,t\right) \tag{2}$$

The often-present assumption that the forward dynamics is affine in the direction of the produced torque yields a slightly constrained representation of the system:

$$\ddot{\boldsymbol{\theta}}(t)=\boldsymbol{f}_1\left(\boldsymbol{\theta}(t),\,\dot{\boldsymbol{\theta}}(t),\,t\right)+\boldsymbol{G}\left(\boldsymbol{\theta}(t),\,\dot{\boldsymbol{\theta}}(t),\,t\right)\boldsymbol{u}(t) \tag{3}$$

It is often useful to rearrange (3) into the standard *(minimal ODE – ordinary differential equation*) form [5]:

$$\boldsymbol{M}\left(\boldsymbol{\theta}(t)\right)\ddot{\boldsymbol{\theta}}(t)+\boldsymbol{N}\left(\boldsymbol{\theta}(t),\dot{\boldsymbol{\theta}}(t)\right)\dot{\boldsymbol{\theta}}(t)+\boldsymbol{R}\left(\boldsymbol{\theta}(t)\right)=\boldsymbol{V}(t)\boldsymbol{u}(t) \tag{4}$$

where $\boldsymbol{M}(\boldsymbol{\theta}(t))$ is the inertia matrix, $\boldsymbol{N}(\boldsymbol{\theta}(t),\dot{\boldsymbol{\theta}}(t))$ describes the influence of centrifugal /Coriolis forces, $\boldsymbol{R}(\boldsymbol{\theta}(t))$ accounts for gravity forces and $\boldsymbol{V}(t)$ is the system input vector.

The system given as (3) or (4) is *fully actuated* in configuration $(\boldsymbol{\theta}(t),\dot{\boldsymbol{\theta}}(t),t)$ if it is able to command immediate acceleration in an arbitrary direction [3]:

$$rank\left(\boldsymbol{G}\left(\boldsymbol{\theta}(t),\,\dot{\boldsymbol{\theta}}(t),\,t\right)\right)=rank\left(\boldsymbol{V}\left(\boldsymbol{\theta}(t)\right)\right)=\dim\left(\boldsymbol{\theta}(t)\right) \tag{5}$$

If the range of directions in which immediate acceleration can be commanded is limited, the system is *underactuated*:

$$rank\left(\boldsymbol{G}\left(\boldsymbol{\theta}(t),\,\dot{\boldsymbol{\theta}}(t),\,t\right)\right)=rank\left(\boldsymbol{V}\left(\boldsymbol{\theta}(t)\right)\right)<\dim\left(\boldsymbol{\theta}(t)\right) \tag{6}$$

Typically, underactuated systems have fewer actuators than DoFs [1]. The difference $\dim(\boldsymbol{\theta}(t))-rank(\boldsymbol{V}(\boldsymbol{\theta}(t)))$ specifies the *degree of underactuation* of the system.

A. *Inverted Pendulum Systems*

Stabilization of a physical pendulum or a system of interconnected pendulum links in the upright unstable position is a benchmark problem in nonlinear control theory: in recent years, several types of stabilizing mechanisms such as cart moving on a rail [5], rotary arm [6], vertical oscillating base, or gyroscope have been introduced. Inverted pendulum systems (IPSs) are therefore regularly employed as typical examples of *unstable nonlinear underactuated systems* in the process of verification of linear or nonlinear control strategies in corresponding control structures [1][5][6]. Direct practical applications include walking robots, launching rockets, earthquake-struck buildings and two-wheel vehicles such as the *Segway PT* [1]. Principles of modeling and control of IPSs can further be considered as the basic starting point for the research of advanced underactuated systems such as mobile robots and manipulators [3], as well as aircraft / watercraft vehicles [2].

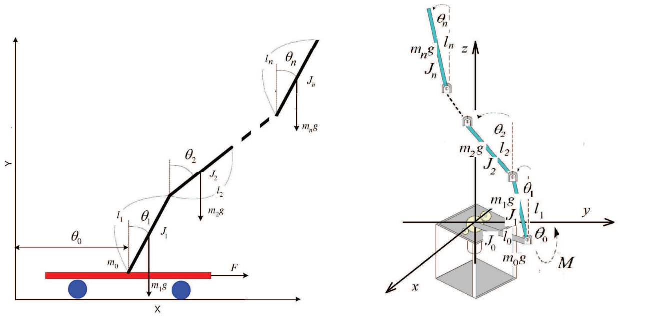

Fig. 1 Scheme and nomenclature: a) Generalized classical inverted pendulum system b) Generalized rotary inverted pendulum system

During our research, we focused on the mutual analogy among mathematical models of IPSs with various number of pendulum links. Consequently, we introduced the concept of a *generalized (n-link) inverted pendulum system* with n+1 DoFs and a single actuator, which allows to treat an arbitrary system of interconnected inverted pendula as a particular instance of the system of n pendula attached to a given stabilizing base (Fig. 1). A general procedure which determines the Euler-Lagrange equations of motion for a user-specified instance of a generalized classical and rotary IPS was developed and implemented via *Symbolic Math Toolbox* [7].

B. *Artificial Underactuated Systems*

Acrobot, *Pendubot* and the *inertia wheel pendulum* are all examples of underactuated systems with two degrees of freedom and a single actuator [2][8] which were introduced artificially to create complex low-order nonlinear dynamics and gain insight into control of high-order underactuated systems. Graphical representation of the *Acrobot* and the *Pendubot* is similar – both systems are depicted as two-link planar robots with revolute joints and share the same matrix of inertia. In the case of *Acrobot*, the actuator is placed at the elbow, while the *Pendubot* is actuated at the shoulder (Fig. 2).

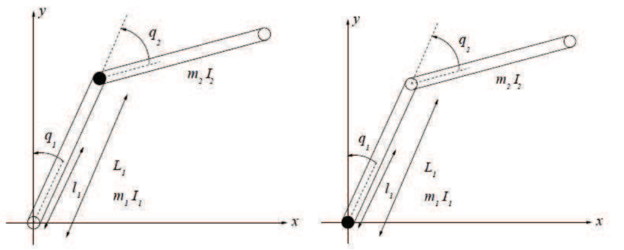

Fig. 2 Scheme and nomenclature: a) Acrobot b) Pendubot

The inertia wheel pendulum is composed of a physical pendulum with a rotating uniform inertia wheel at the end of the pendulum rod which is not directly actuated: in order to stabilize the pendulum in the upright equilibrium, the system has to be controlled via the rotating wheel.

C. *Mobile and Manipulator Robotic Systems*

Stabilization and tracking problems in *mobile robotics* generally involve underactuated mechanical systems [2][8]. If a robot with n inner connections and n actuators is not attached to the ground and instead performs walking, brachiating, gymnastic, swimming or flying motion [2], the number of its DoFs increases by the six DoFs which define its position and spatial orientation. Every additional control surface (i.e. a moveable platform) adds another actuator and a DoF to the system. *Robot manipulators* are often underactuated by construction, and a fully actuated manipulator becomes underactuated whenever the manipulated body provides the system with additional DoFs.

Principles of control for *underactuated systems* can also be employed to improve control of *fully actuated systems* either by increasing the effectivity of the employed actuators or by decreasing the design complexity [3]. After all, if the standard initial assumption of *rigid robotic arms* is omitted, we can claim that every robotic system is underactuated.

D. *Aircraft & Watercraft Systems*

Two significant groups of underactuated systems include aircraft (helicopters, airplanes, spaceships, satellites) and watercraft systems (ships, boats, submarines). Stabilization of the system in the direction of individual DoFs in the water/air environment, trajectory tracking and planning are specified as the principal analyzed problems. Underactuation is generally implied by the design and construction of a particular vehicle (Fig. 3). The *PVTOL* (*planar vertical take-off and landing*) airplane system is an underactuated system with three DoFs and two actuators which is often employed as a simplified planar model of the takeoff and landing of a helicopter [8]. The *helicopter/airplane system* is standardly described by six DoFs – position (x, y, z) and rotation angles along the three axes (*pitch* – lateral, *roll* – longitudinal, *yaw* – vertical rotation) and four control inputs – three control moments in the body frame and the main rotor thrust [2]. Ocean vessels are equipped with propellers and rudders which enable control in two directions only (*surge* – longitudinal, *heave* – vertical axis) without any direct control in the direction of lateral motion (*sway*) [9]. The degree of underactuation can also increase in the case of actuator failure, or if the number of motors is intentionally reduced to decrease the load mass onboard the aircraft or watercraft.

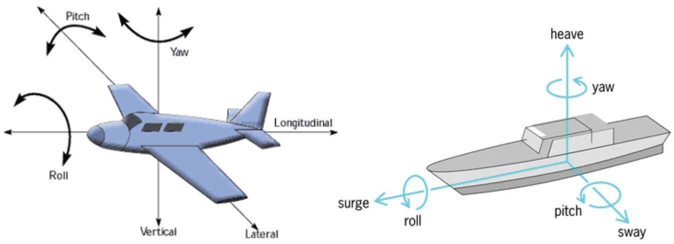

Fig. 3 Scheme and degrees of freedom related to principal axes: a) Airplane b) Ship

Living systems are often underactuated in the interaction with their surroundings. Although the human body contains more *actuators* (muscles) than *DoFs* (joints), it can be easily proven that our body is underactuated despite the presence of fully actuated joints: if we jump into air, no combination of control inputs from muscles is able to counter the influence of gravity and aerodynamic forces and change the trajectory of our center of gravity [3]. Similar considerations apply to the other living creatures. A flying bird, a swimming fish and a walking human are all examples of mechanical systems whose locomotion is due to changes in their physical shape, leading to indirect position actuation. As it is shown in [10], by studying various types of animal locomotion and the way they overcome their own underactuated dynamics, we can get significant inspiration for design of underactuated vehicles.

III. Optimal Control of Nonlinear Underactuated Systems

Fully actuated systems possess a number of strong structural properties which facilitate design of optimal/robust/ adaptive controllers, e.g. feedback linearizability, passivity and linear parametrizability. These are usually lost in underactuated systems, while at the same time undesirable properties (higher relative degree, nonminimum phase behavior) emerge [2]. Control of underactuated systems subsequently becomes more difficult, with fewer general results available.

It has been shown that optimal control techniques yield reliable, consistent results for underactuated systems. The goal of optimal control design for a *linear, time-invariant* dynamic system is to determine such feedback control so that a given criterion of optimality is achieved [11]. In case the considered linear system is actually a linear approximation of a nonlinear system around a given equilibrium, then the optimal techniques for linear systems yield an approximate, locally near-optimal stabilizing solution with guaranteed closed-loop stability and robustness. In [1][12], we solved the problem of IPSs stabilization in the unstable position via optimal control algorithms based on quadratic functional minimization, using continuous-time and discrete-time linearized state-space models of IPSs. We also introduced additional control structures which ensure that the cart/arm position reaches the reference value (by means of feedforward gain), and the permanent steady-state error is eliminated (by implemented integral control).

Model predictive control (MPC) is a discrete-time optimal control technique in which the control action for each time step is computed by solving an on-line optimization problem in finite time (*receding horizon control*) while at the same time considering input/state constraints [13]. MPC is currently the only modern control technique with significant impact on industrial process control: compared to the 1980s, when MPC technology first became popular in petrochemical industry, commercial MPC implementations can now be found in chemical and food processing, automotive and aerospace applications [15]. However, application of the MPC algorithm for most nonlinear underactuated systems still presents a challenge in terms of disturbance/steady-state error elimination. To solve problems arising from the structure of an underactuated system, suitable adjustment of the MPC algorithm [13][14] is required.

IV. Hybrid Systems Theory in Modeling and Control of Underactuated Systems

To provide a convenient framework for modeling and control of systems characterized by an interaction between continuous (*time-driven*) and discrete (*event-driven*) dynamics, *hybrid systems theory* was developed [16]. As a result, various engineering problems which were once considered a case of a particular implementation can now be researched systematically as part of a complex theory.

A. Modeling of Underactuated Hybrid Systems

Hybrid models are often useful if we have to consider discontinuous development of the mechanical system dynamics, i.e. a robotic arm whose continuous motion is interrupted by collisions or strikes to the surface, or if the arm dynamics is subject to state jumps caused by the arm shooting out objects. In [17], a hybrid model of such an underactuated robot is defined as a modification of the standard minimal form (which includes an operator describing the jumps in the state vector), and employed in a trajectory planning algorithm. Out of the available mathematical formalisms (modeling frameworks) of hybrid systems, *PWA* and *MLD* forms are often employed: the *PWA* form interconnects the linear state-space representation and discrete automata, dividing the input/state space into regions defined by polyhedra, and the *MLD* form is composed of a system of linear difference equations which can assume real and binary values and a set of linear inequalities to describe the constraints [18].

B. Control of Underactuated Hybrid Systems

Optimal control theory is the principal approach to hybrid systems control, and the complexity of an optimal control problem decreases if the system is expressed in discrete-time, since its main source in a hybrid system is the number of possible switching scenarios. Optimal control problems can be solved for hybrid systems in the discrete-time state-space form using either *PWA* or *MLD* models, which was first outlined by Sontag in [19]. It is next shown by Borrelli [18] that the solution of the optimal control problem in finite time is a time-variant piecewise-affine feedback control law, defined over non-convex regions. *Hybrid model predictive control*, which has recently attracted much attention, is suitable for systems defined by switched linear dynamics which are subject to linear/logical constraints on state/input variables [20].

Application of hybrid optimal/predictive control algorithms on underactuated systems has already been covered by multiple authors, although no consistent approach has yet been developed. In a survey paper, Buse et al. [21] demonstrate the application of hybrid optimal control techniques in control of an underactuated robot arm. Yin & Hosoe [22] employ hybrid predictive control to plan the trajectory of a walking robot expressed in *MLD* form, and Rodrigues & How [23] develop an algorithm to automate the transformation of IPSs into the *PWA* representation to enable subsequent hybrid control.

The advancement of hybrid systems theory is supported by a wide range of available software tools which enable symbolic/numeric computations and simulations in accordance with theoretical results. Most of these have been developed by research groups led by *Prof. Morari* of ETH Zürich and *Prof. Bemporad* of ETH Zürich, later University of Siena [18][20], and include tools which simplify the process of formulation and analysis of a hybrid model (*HYSDEL* modeling language), enable experimental identification of hybrid models (*Hybrid Identification Toolbox* (HIT)), and provide functions for hybrid optimal/predictive control algorithm design as well as closed-loop simulation (*Hybrid Toolbox* (HT) [24], and *Multiparametric Toolbox* (MPT) [25]).

From a different viewpoint, Liberzon [26] presents examples of control problems of underactuated systems where it is necessary or useful to employ switching control structures. If the desired trajectory of the underactuated system consists of multiple pieces of significantly different parts (e.g. aircraft maneuvers) or if the state space contains obstacles, we might need to choose different controllers at different stages of the problem and implement switching between them. Also, switching control is often the only feasible way to control a nonholonomic system, since there is no continuous control which could stabilize such systems on a given time interval.

V. Conclusion

This paper presents a compact summary of results which have so far been achieved in modeling and optimal control of nonlinear mechanical underactuated systems using classical and hybrid approaches. Great practical importance of underactuated systems in mobile robotics, aviation and ship transport has sparked much interest from physicists and control theorists alike. The principal aim of currently conducted research is to overcome the difficulty of control algorithm design caused by certain disadvantageous physical properties of underactuated systems.

After a brief survey on fundamental principles of mechanical system modeling based on Lagrangian mechanics, an overview of principal categories of underactuated systems was presented. For each category, the reason for underactuation was specified together with control objectives addressed in referenced works. Optimal control was confirmed as a reliable control technique for underactuated systems; it was shown that adjustments to predictive algorithms are required. Application areas of hybrid systems theory were discovered to include hybrid models for underactuated systems with logical parts, as well as hybrid optimal/predictive control algorithms and switching control structures. Numerous problems with future research potential were emphasized in the paper.

The findings presented in this paper are elaborated in the referenced thesis for dissertation examination which describes the proposed integration of theories of underactuated systems, optimal control, and hybrid systems. As a meaningful contribution to modeling/control education, nonlinear underactuated systems are being integrated into the research and teaching activities of the Center of Modern Control Techniques and Industrial Informatics at the DCAI-FEEI TU.

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