ELEKTRO 2012

A Complex Overview of the Rotary Single Inverted Pendulum System

13. 10. 2012

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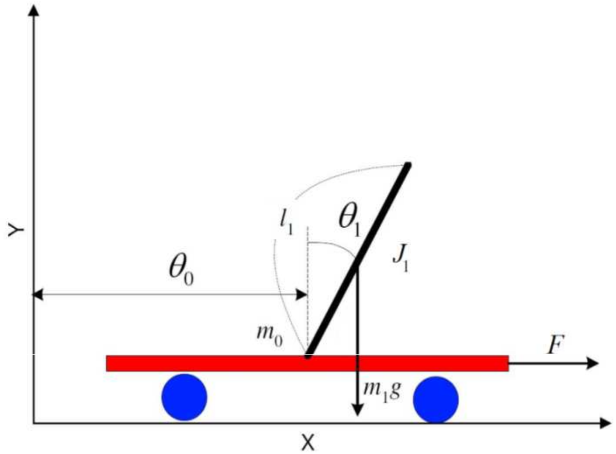
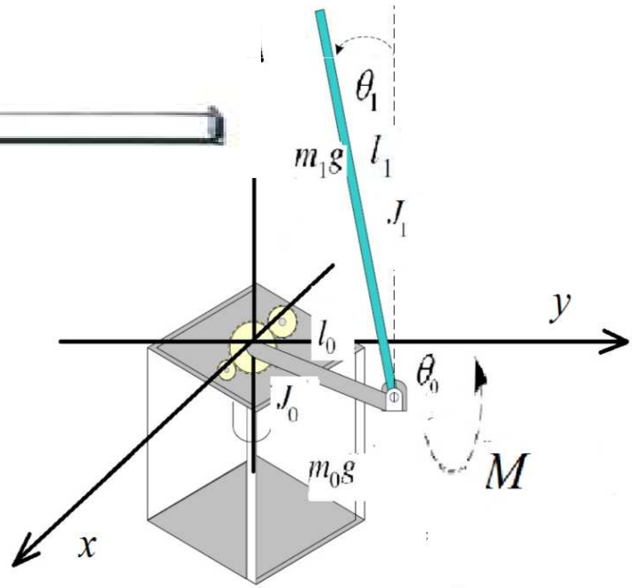
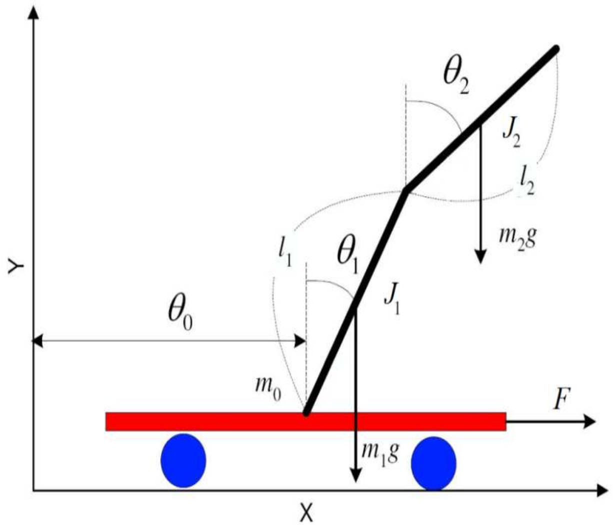
Inverted Pendula Systems – a class of mechanical systems significant for control theory







Main Points of the Presented Problem

I. **mathematical modeling and simulation of the rotary inverted pendulum system**
   - automatic derivation of motion equations
   - open-loop dynamical analysis

II. **stabilization of the rotary single inverted pendulum via state-feedback control techniques**
   - automatic linear approximation of the system
   - state-feedback control with a state estimator
   - state-feedback control with permanent disturbance compensation

III. **conclusion and evaluation of the achieved results**

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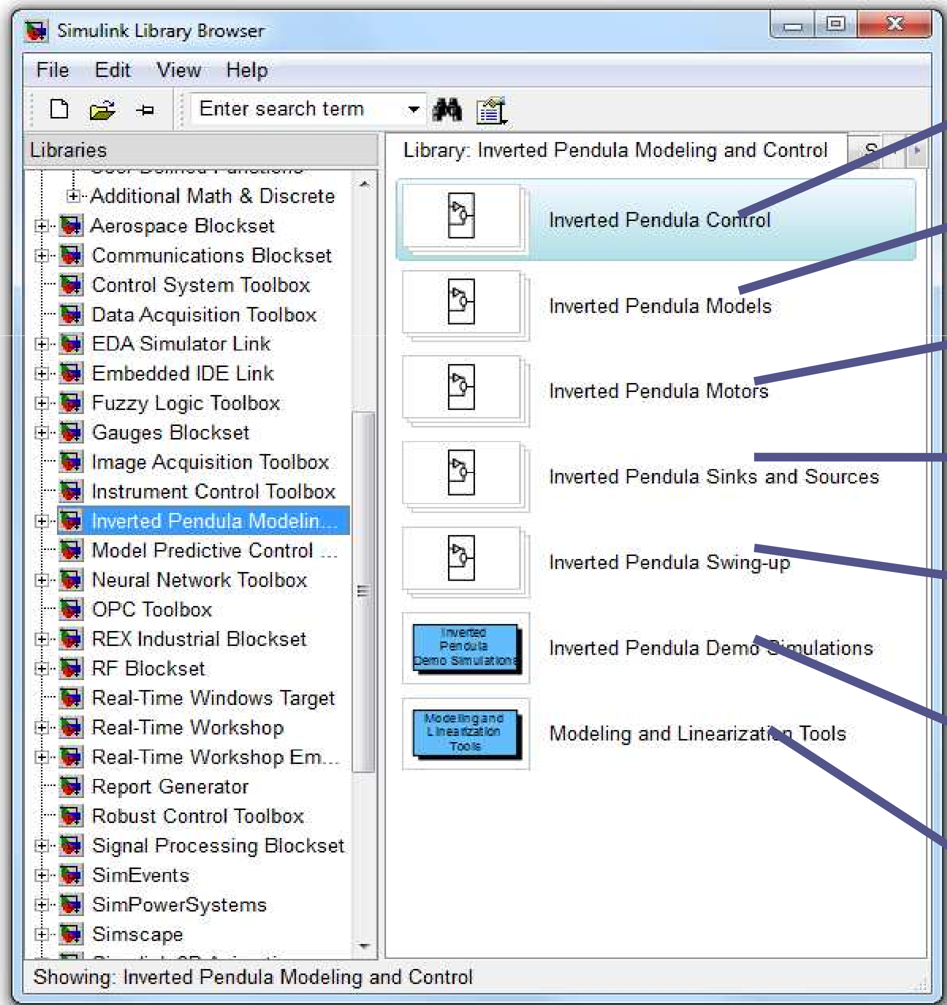
Inverted Pendula Modeling and Control - IPMaC (Simulink block library)



- state-feedback control
- simulation models of inverted pendula
- direct-current motor
- input/output blocks
- swing-up into the upright position
- demonstrations of the block functionality
- GUI applications

A.
Mathematical Modeling and Simulation of the Rotary Single Inverted Pendulum

A.

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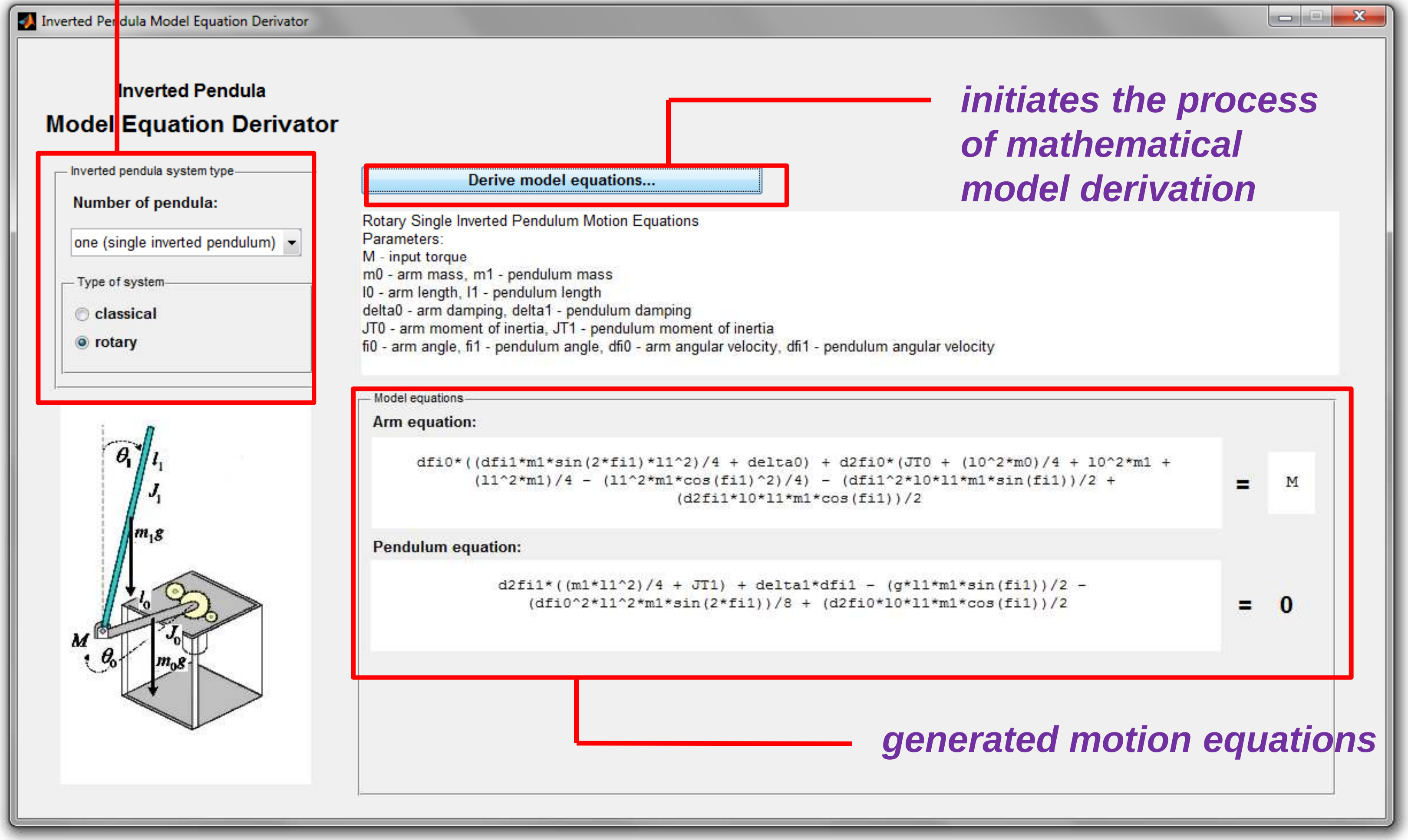
Inverted Pendula Model Equation Derivator (automatic derivation of motion equations)

selection of system type & number of pendulum links

initiates the process of mathematical model derivation

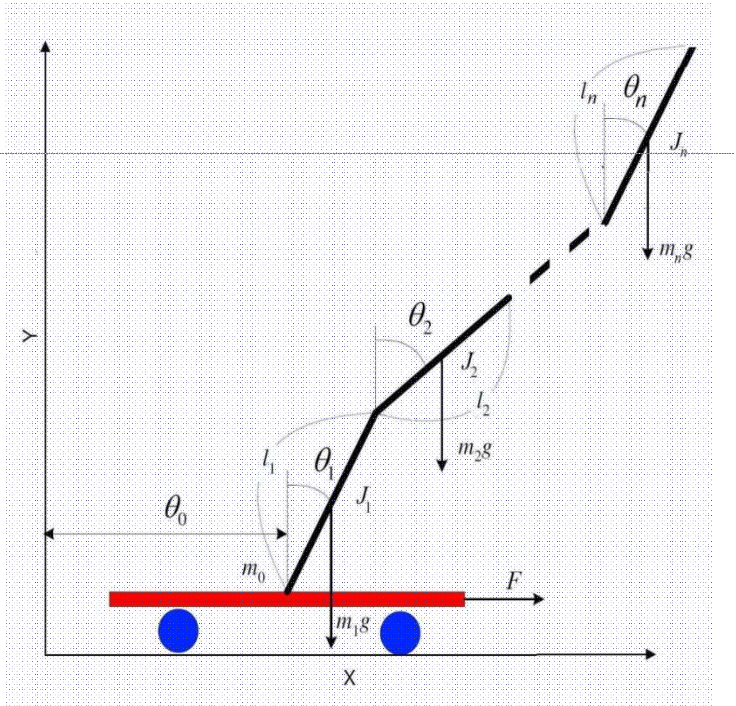


generated motion equations

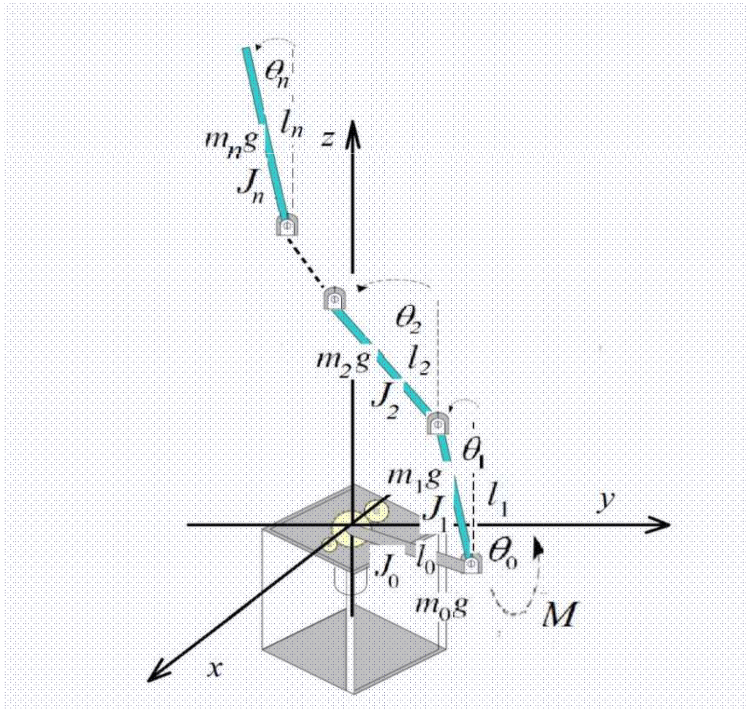
A.
Generalized approach to inverted pendula modeling

System description: $\boldsymbol{\theta}(t) = (\theta_0(t) \quad \theta_1(t) \quad \dots \quad \theta_n(t))^T$

cart/arm position — pendula angles



system of *n* classical inverted pendula



system of *n* rotary inverted pendula

A.

General procedure of mathematical model derivation for inverted pendula systems - brief outline

Vector of generalized coordinates:

$$\boldsymbol{\theta}(t) = \left(\theta_0(t) \quad \theta_1(t) \quad \ldots \quad \theta_n(t)\right)^T$$

Lagrange equations of the second kind:

$$\frac{d}{dt}\left(\frac{\partial L(t)}{\partial \dot{\boldsymbol{\theta}}(t)}\right) - \frac{\partial L(t)}{\partial \boldsymbol{\theta}(t)} + \frac{\partial D(t)}{\partial \dot{\boldsymbol{\theta}}(t)} = \mathbf{Q}^*(t)$$

`invpenderiv.m`

`rotinvpenderiv.m`

MATLAB functions which derive the motion equations for a system with a given number of pendulum links

Rearrangement into standard minimal ODE form:

$$\boldsymbol{M}\left(\boldsymbol{\theta}(t)\right)\ddot{\boldsymbol{\theta}}(t) + \boldsymbol{N}\left(\boldsymbol{\theta}(t),\dot{\boldsymbol{\theta}}(t)\right)\dot{\boldsymbol{\theta}}(t) + \boldsymbol{P}\left(\boldsymbol{\theta}(t)\right) = \boldsymbol{V}(t)$$

A.

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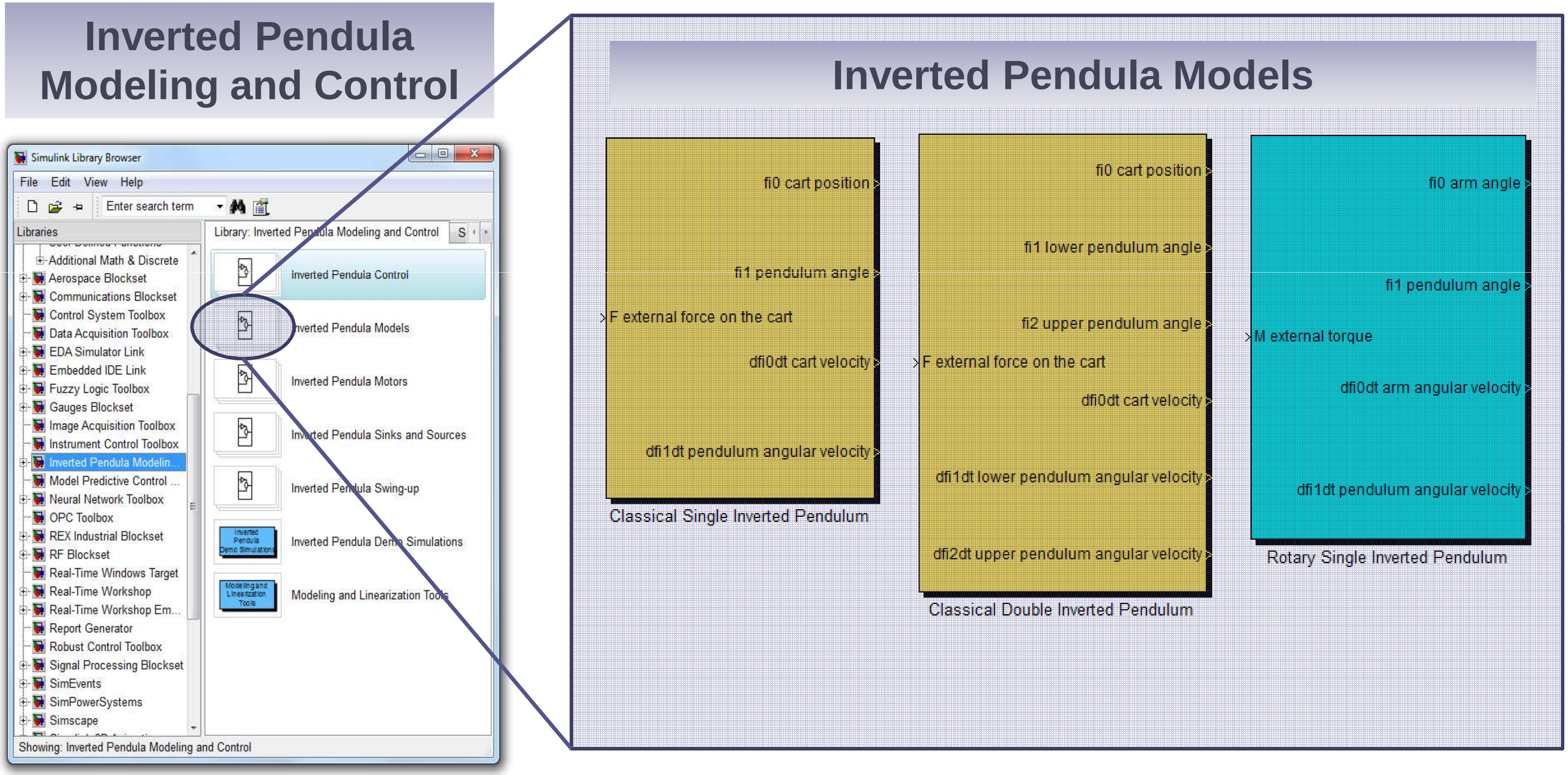
Simulation models of selected inverted pendula systems (dynamic-masked *Simulink* library blocks)

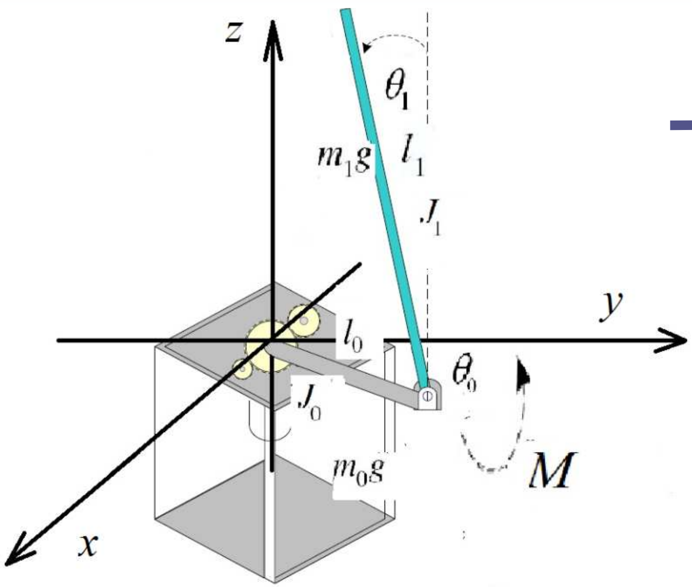


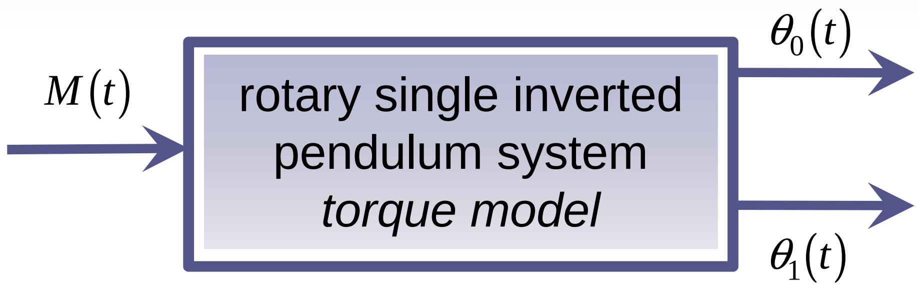
A.

Rotary single inverted pendulum system - scheme and generated mathematical model

$$\begin{pmatrix} J_0 + m_1 l_0^2 + \frac{1}{4} m_1 l_1^2 \sin^2\theta_1(t) & \frac{1}{2} m_1 l_0 l_1 \cos\theta_1(t) \\ \frac{1}{2} m_1 l_0 l_1 \cos\theta_1(t) & J_1 \end{pmatrix} \begin{pmatrix} \ddot{\theta}_0(t) \\ \ddot{\theta}_1(t) \end{pmatrix} + \begin{pmatrix} \delta_0 + \frac{1}{4} m_1 l_1^2 \dot{\theta}_1(t) \sin 2\theta_1(t) & -\frac{1}{2} m_1 l_0 l_1 \dot{\theta}_1(t) \sin\theta_1(t) \\ -\frac{1}{8} m_1 l_1^2 \dot{\theta}_0(t) \sin 2\theta_1(t) & \delta_1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_0(t) \\ \dot{\theta}_1(t) \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 \\ -\frac{1}{2} m_1 g l_1 \sin\theta_1(t) \end{pmatrix} = \begin{pmatrix} M(t) \\ 0 \end{pmatrix}$$

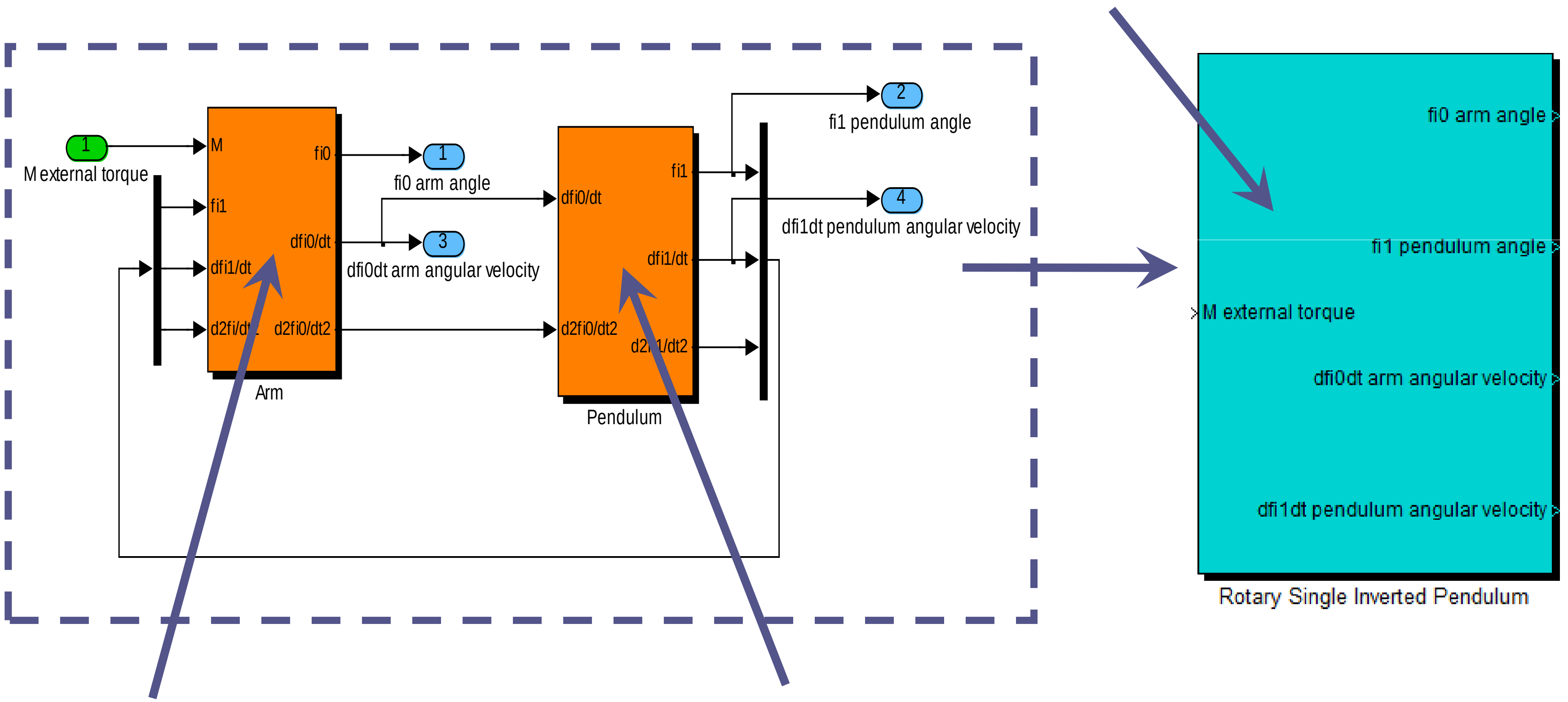




$$m_0 = 0{,}5kg \quad m_1 = 0{,}275kg$$
$$l_0 = 0{,}6m \quad l_1 = 0{,}5m$$
$$\delta_0 = 0{,}3kgs^{-1} \quad \delta_1 = 0{,}011458s^{-1}$$

A.
Rotary single inverted pendulum system - library block structure

Rotary Single Inverted Pendulum function block



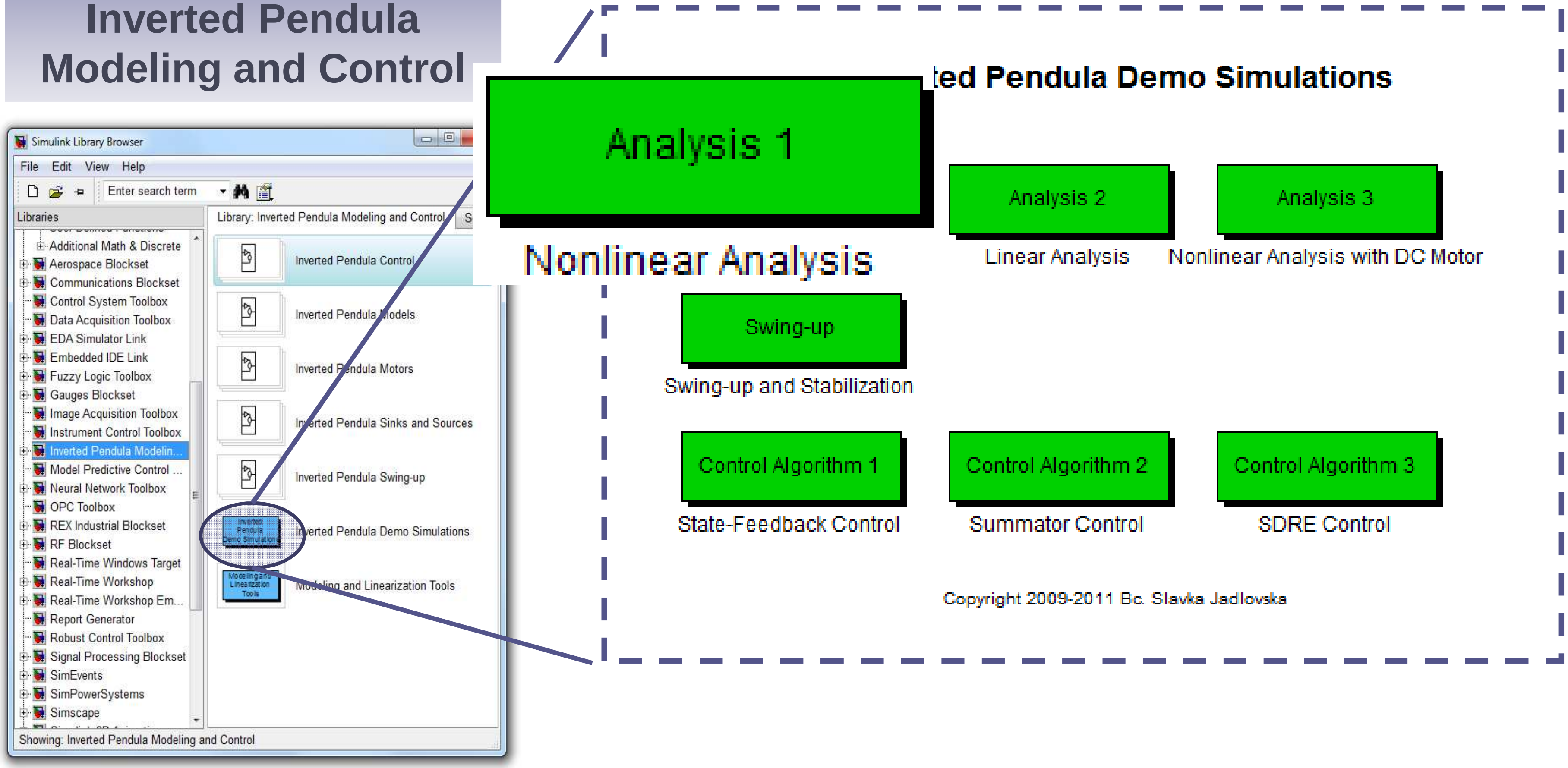
rotary arm subsystem

pendulum subsystem

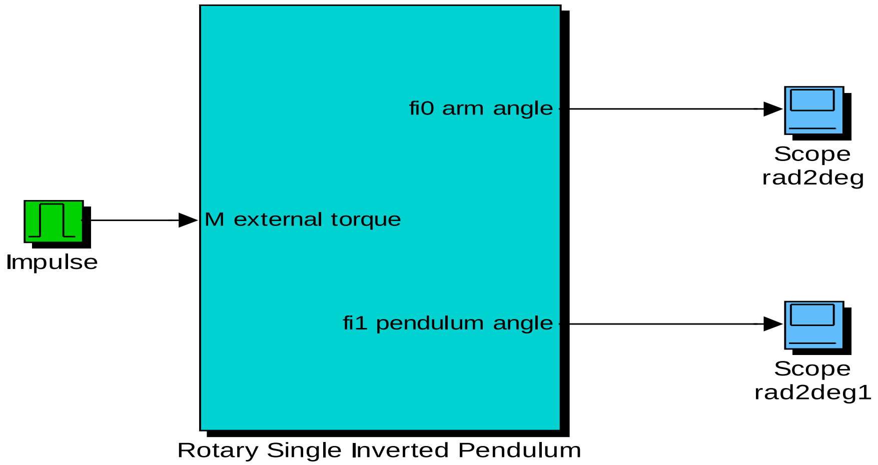
A.

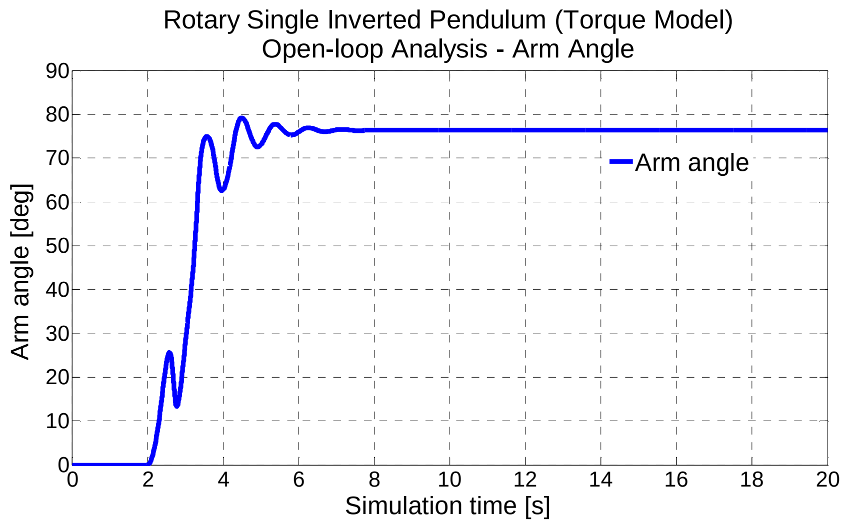
Demo Simulations I: Open-loop dynamical analysis for nonlinear force-torque models of inverted pendula systems

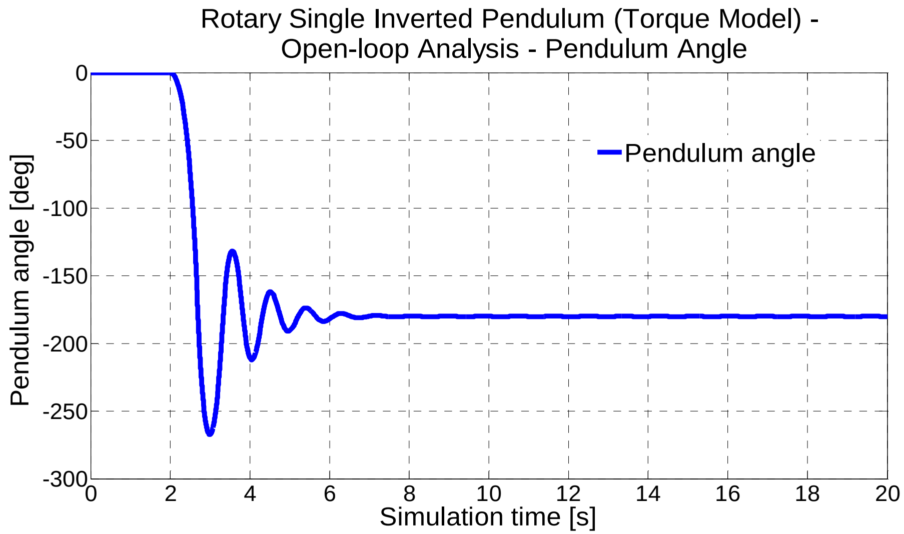
Inverted Pendula Modeling and Control



A. Rotary single inverted pendulum (torque model) - open-loop dynamical analysis







A.

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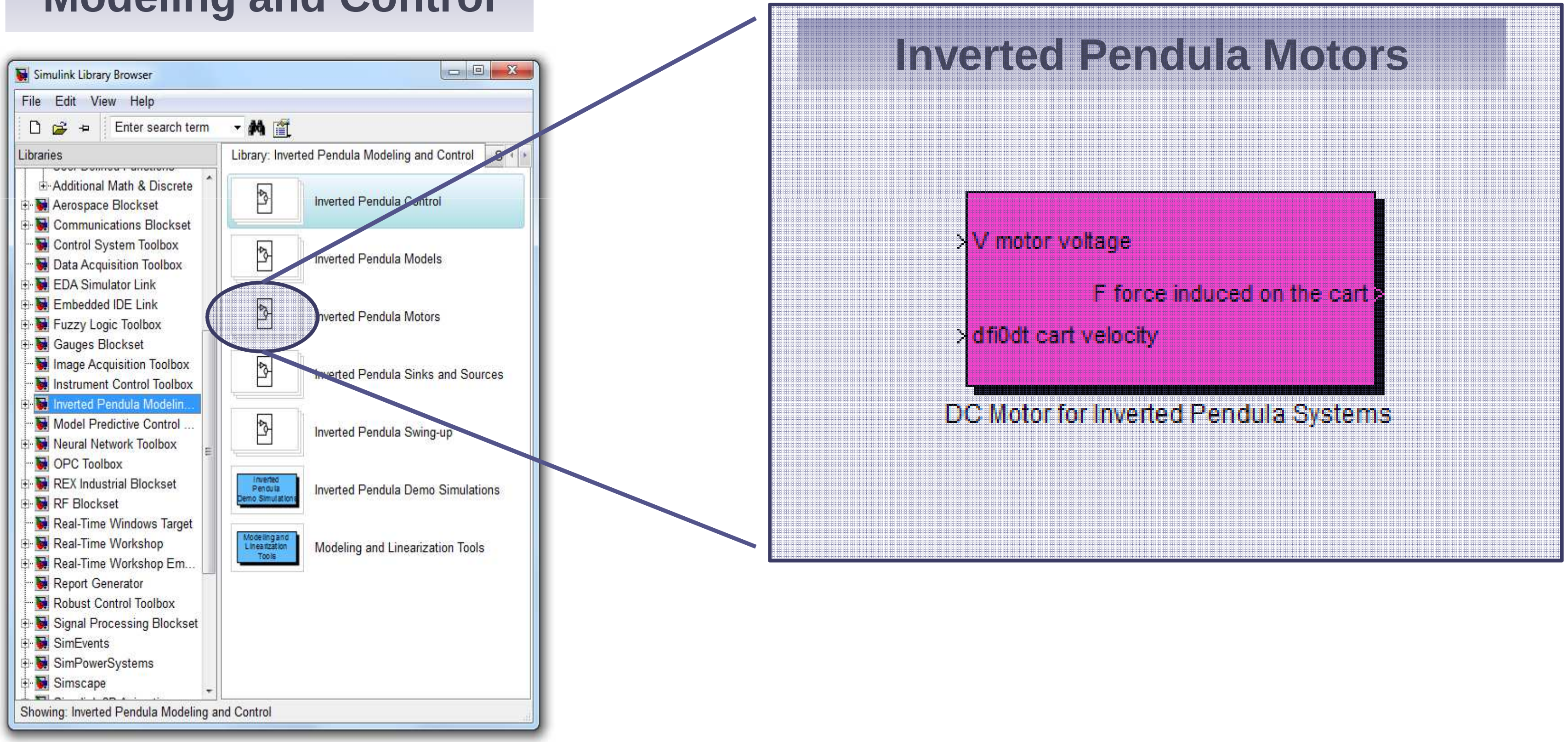
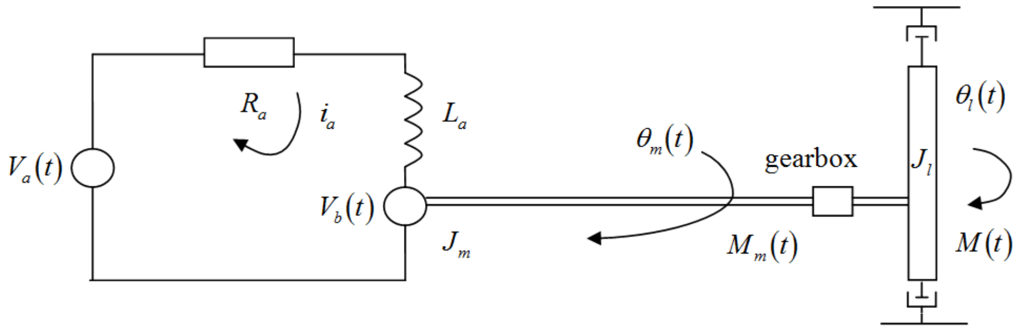
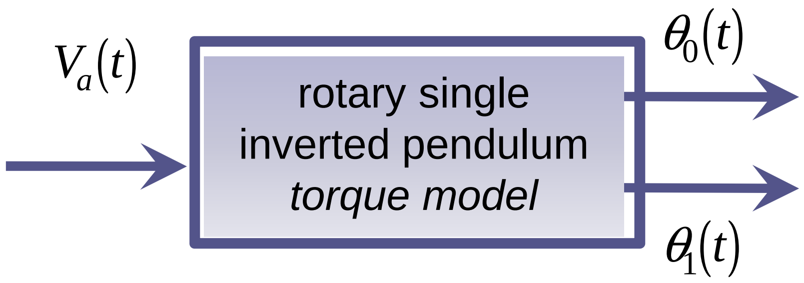
Simulation model of actuating mechanism (DC motor) for inverted pendula systems (*Simulink* library block)

Inverted Pendula Modeling and Control



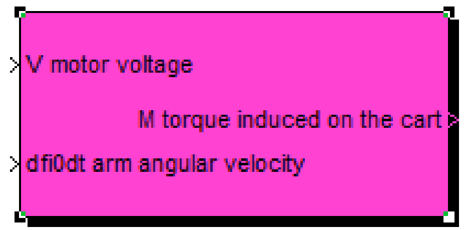
A. *DC motor model* in form of a voltage-to-torque conversion relationship





rotary single inverted pendulum system:

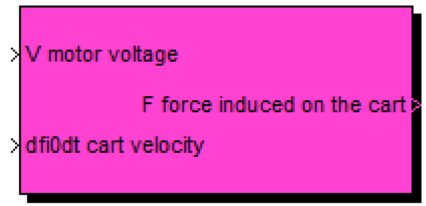
$$M(t) = \frac{k_m k_g}{R_a} V_a(t) - \frac{k_m^2 k_g^2}{R_a} {}^r\dot{\theta}_0(t)$$



DC Motor for Inverted Pendula Systems

classical single inverted pendulum system

$$F(t) = \frac{k_m k_g}{R_a r} V_a(t) - \frac{k_m^2 k_g^2}{R_a r^2} {}^c\dot{\theta}_0(t)$$



DC Motor for Inverted Pendula Systems

A.

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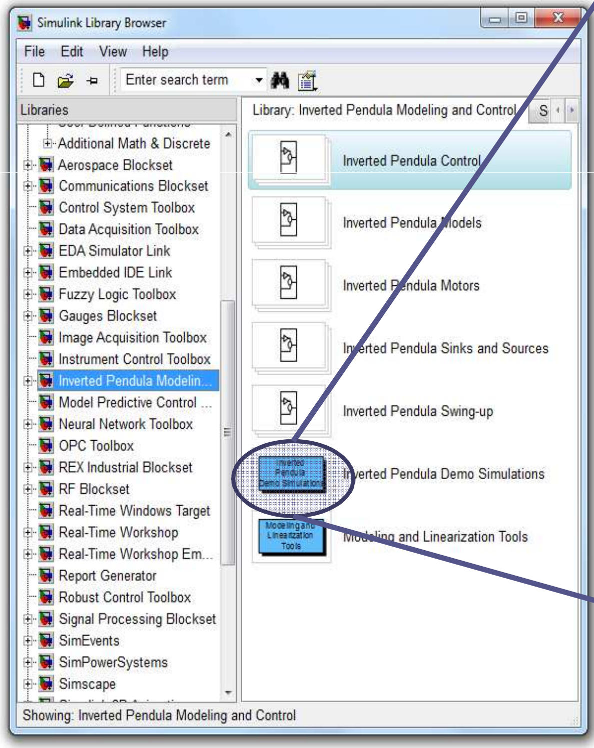
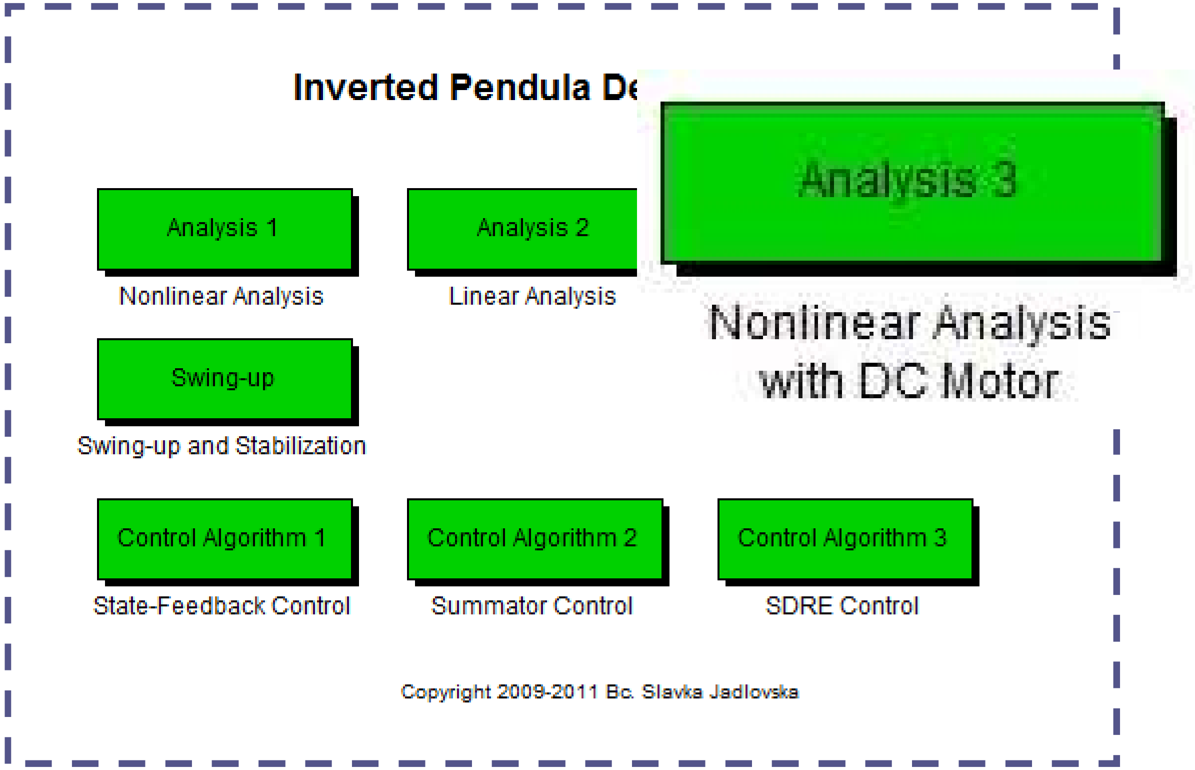
Demo Simulations II: Open-loop dynamical analysis for nonlinear voltage models of inverted pendula systems

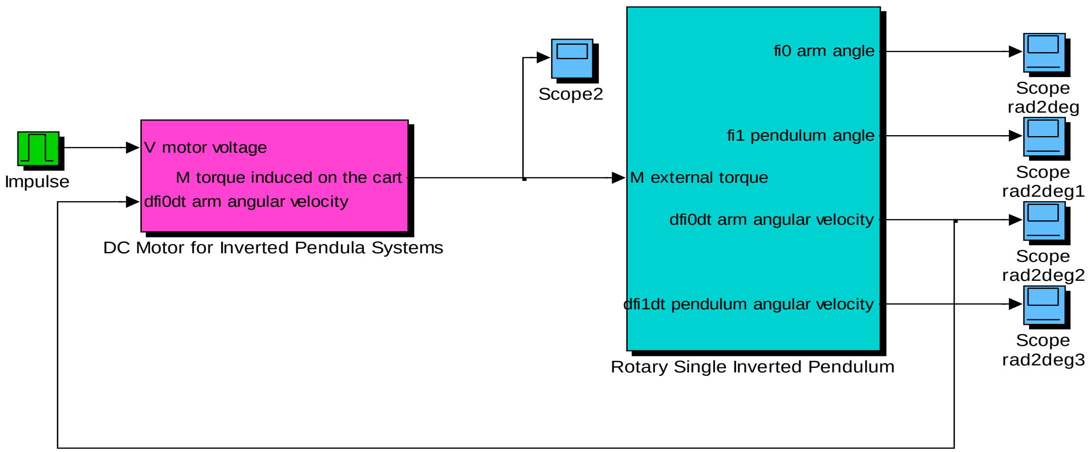
Inverted Pendula Modeling and Control

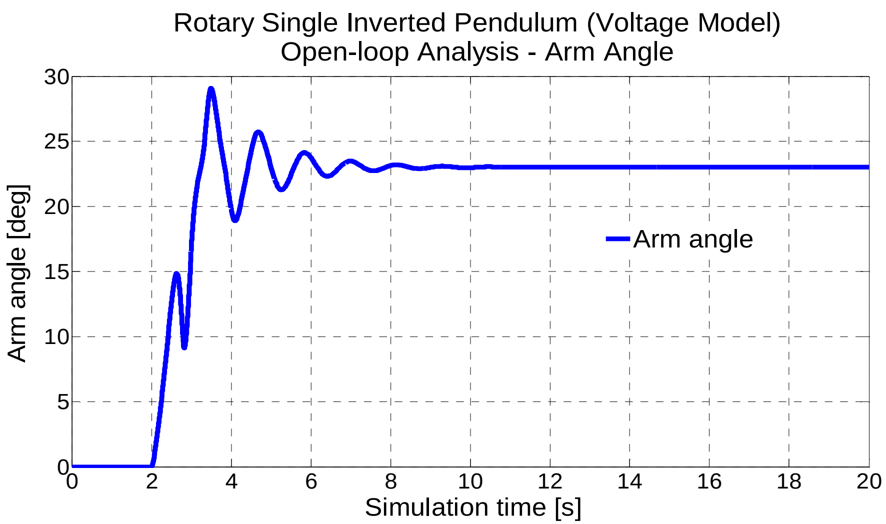


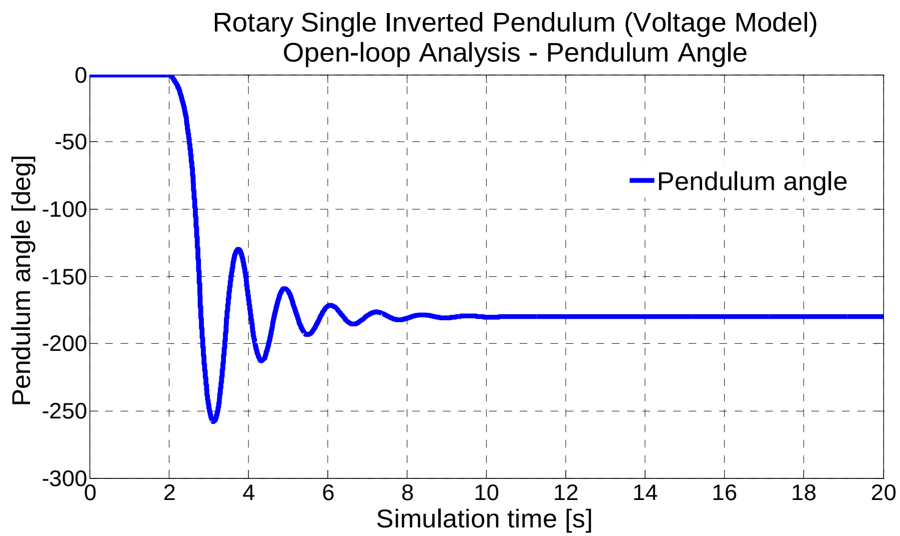


A.

Rotary single inverted pendulum (voltage model) - open-loop dynamical analysis







B.
Stabilization of the Rotary Single Inverted Pendulum via State-Feedback Control Techniques

B.

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Control techniques for inverted pendula systems supported by the *IPMaC* block library

IPMaC

Control Objective

Derivator

Stabilization of the pendulum in the upright position

Inverted Pendula Models

Swing-up into the upright position

Linearizator

Linear state-space pendulum model

- pole-placement algorithm
- linear quadratic optimal control method (LQR)
- *PID/PSD algorithm*

Nonlinear state-space pendulum model

- state-dependent Riccati equation
- *input-output linearization*
- *artificial intelligence*

Nonlinear state-space pendulum model

- energy-based methods
- *artificial intelligence*

Inverted Pendula Control

Inverted Pendula Swing-up

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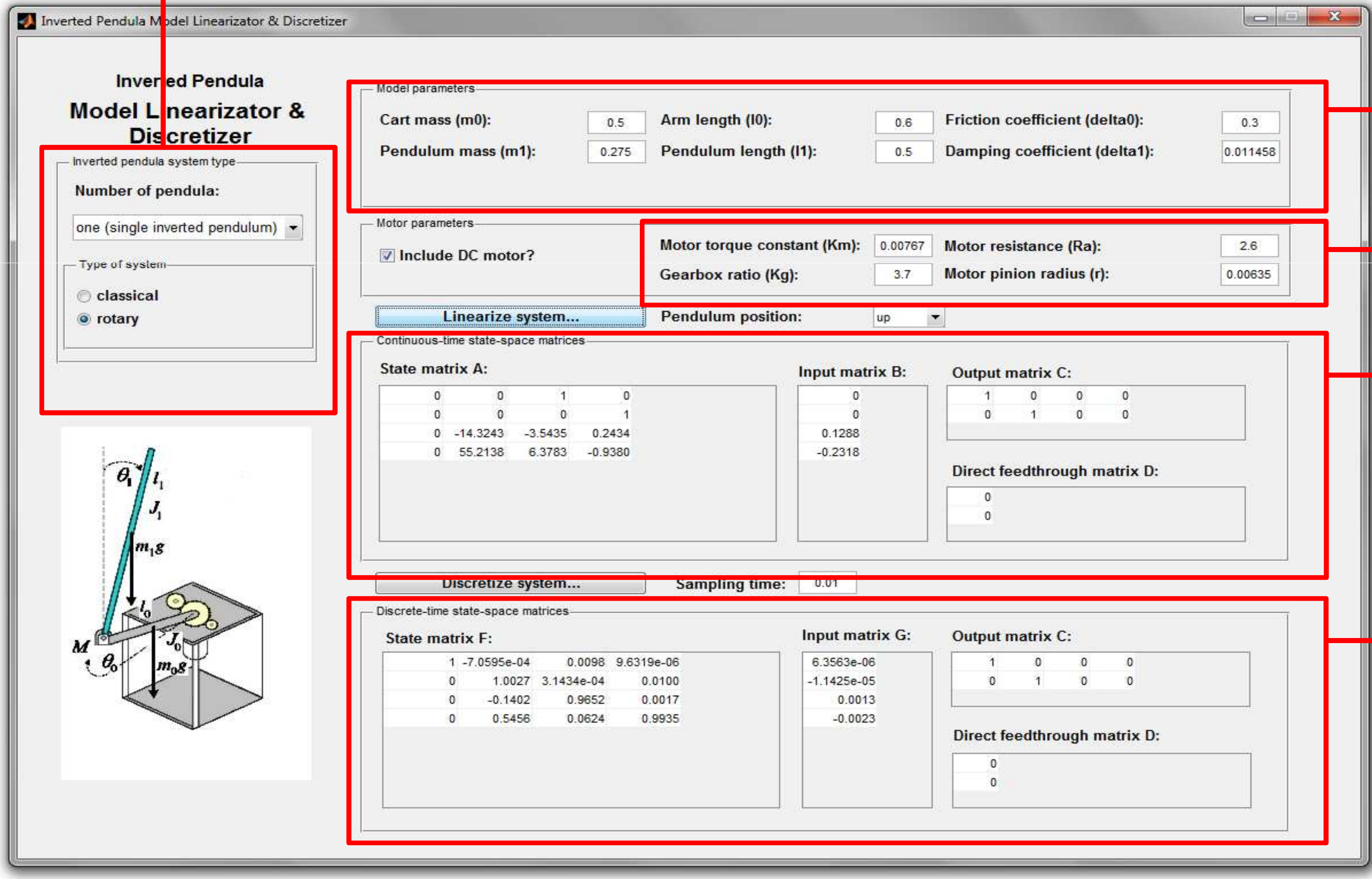
B. 1)
Inverted Pendula Model Linearizator & Discretizer

selection of system type & number of pendulum links



system parameters

motor parameters

matrices of the linearized system

matrices of the discretized system

B. 1)
State-space description of inverted pendula systems

$$\boldsymbol{x}(t)=\left(\theta(t)\quad \dot{\theta}(t)\right)^T$$

state vector

Standard minimal ODE form

$$\boldsymbol{M}(\theta(t))\ddot{\theta}(t)+\boldsymbol{N}\left(\theta(t),\dot{\theta}(t)\right)\dot{\theta}(t)+\boldsymbol{P}(\theta(t))=\boldsymbol{V}(t)$$

$$\ddot{\theta}(t)=\left(\boldsymbol{M}(\theta(t))\right)^{-1}\left(V(t)-\boldsymbol{N}\left(\theta(t),\dot{\theta}(t)\right)\dot{\theta}(t)+\boldsymbol{P}(\theta(t))\right)$$

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{f}\left(\boldsymbol{x}(t),u(t),t\right)$$

$$\boldsymbol{y}(t)=\boldsymbol{g}\left(\boldsymbol{x}(t),u(t),t\right)$$

nonlinear state-space description

B. 1) Linear approximation of inverted pendula systems (upright position)

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$$\mathbf{x}(t) = \left(\boldsymbol{\theta}(t) \quad \dot{\boldsymbol{\theta}}(t)\right)^T$$

state vector

$$\dot{\mathbf{x}}(t) = \boldsymbol{f}\left(\mathbf{x}(t), u(t), t\right)$$
$$\boldsymbol{y}(t) = \boldsymbol{g}\left(\mathbf{x}(t), u(t), t\right)$$

$$\mathbf{x}_S = \mathbf{0}^T$$

$$f_i^*\left(\mathbf{x}(t), u(t)\right) \approx f_i\left(\mathbf{x}_S, u_S\right) + \sum_{k=1}^{2n+2} \left.\frac{\partial f_i\left(\mathbf{x}(t), u(t)\right)}{\partial x_k(t)}\right|_{\mathbf{x}_S, u_S} \left(x_k(t) - x_{kS}\right) + \left.\frac{\partial f_i\left(\mathbf{x}(t), u(t)\right)}{\partial u(t)}\right|_{\mathbf{x}_S, u_S} \left(u(t) - u_S\right)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)$$
$$y(t) = \mathbf{C}x(t) + du(t)$$

matrices_single.m

matrices_double.m

matrices_rotary.m

B. 1)

Linearized (continuous-time) and discretized (discrete-time) state-space description of rotary single inverted pendulum

```
[A,B,C,D] = matrices_rotary([0.5 0.6 0.275 0.5
0.3 0.011458],'up','m')
```

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -14{,}3243 & -3{,}5435 & 0{,}2434 \\ 0 & 55{,}2138 & 6{,}3783 & -0{,}938 \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 0{,}1288 \\ -0{,}2318 \end{pmatrix} \quad \boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \boldsymbol{d} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

```
Ts=0.01
[F,G]=c2d(A,B,Ts)
```

$$\boldsymbol{F} = \begin{pmatrix} 1 & -0{,}0007 & 0{,}0098 & 0 \\ 0 & 1{,}0027 & 0{,}0004 & 0{,}01 \\ 0 & -0{,}1402 & 0{,}9652 & 0{,}0017 \\ 0 & 0{,}5456 & 0{,}0624 & 0{,}9935 \end{pmatrix} \quad \boldsymbol{g} = \begin{pmatrix} 0{,}000006 \\ -0{,}000011 \\ 0{,}0013 \\ -0{,}0023 \end{pmatrix} \quad \boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \boldsymbol{d} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

B. 2)

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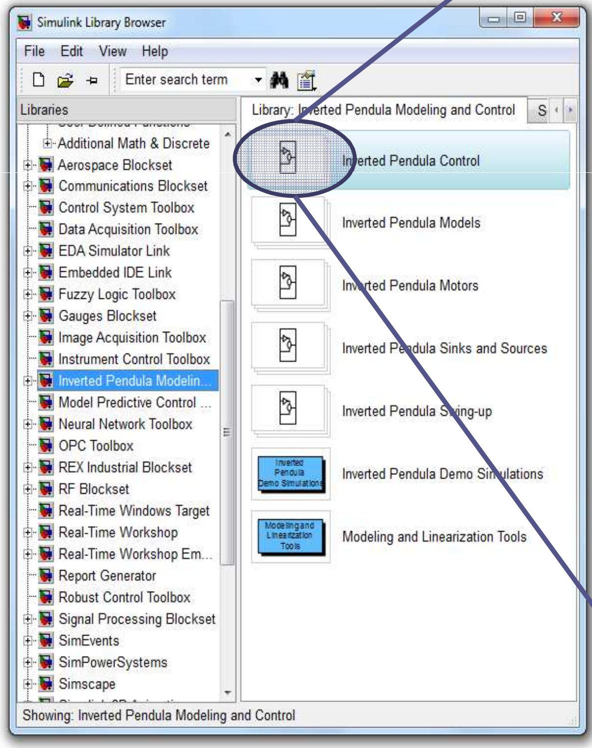
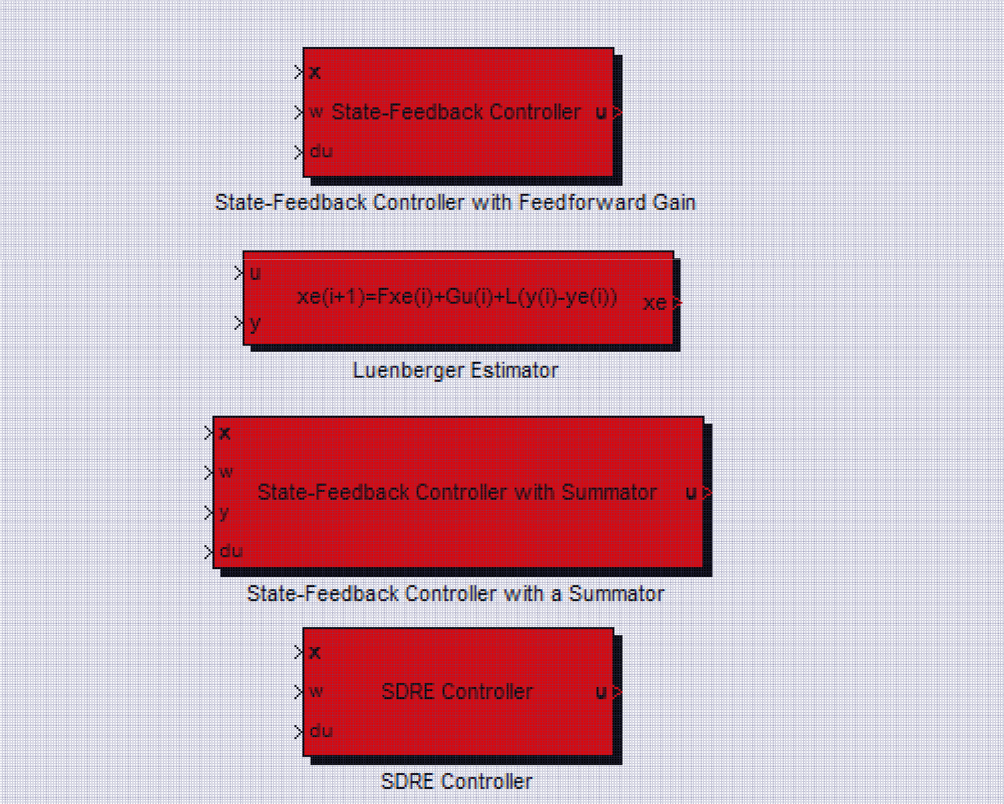
Software support for state-feedback controller design (*Simulink* library blocks)

Inverted Pendula Modeling and Control

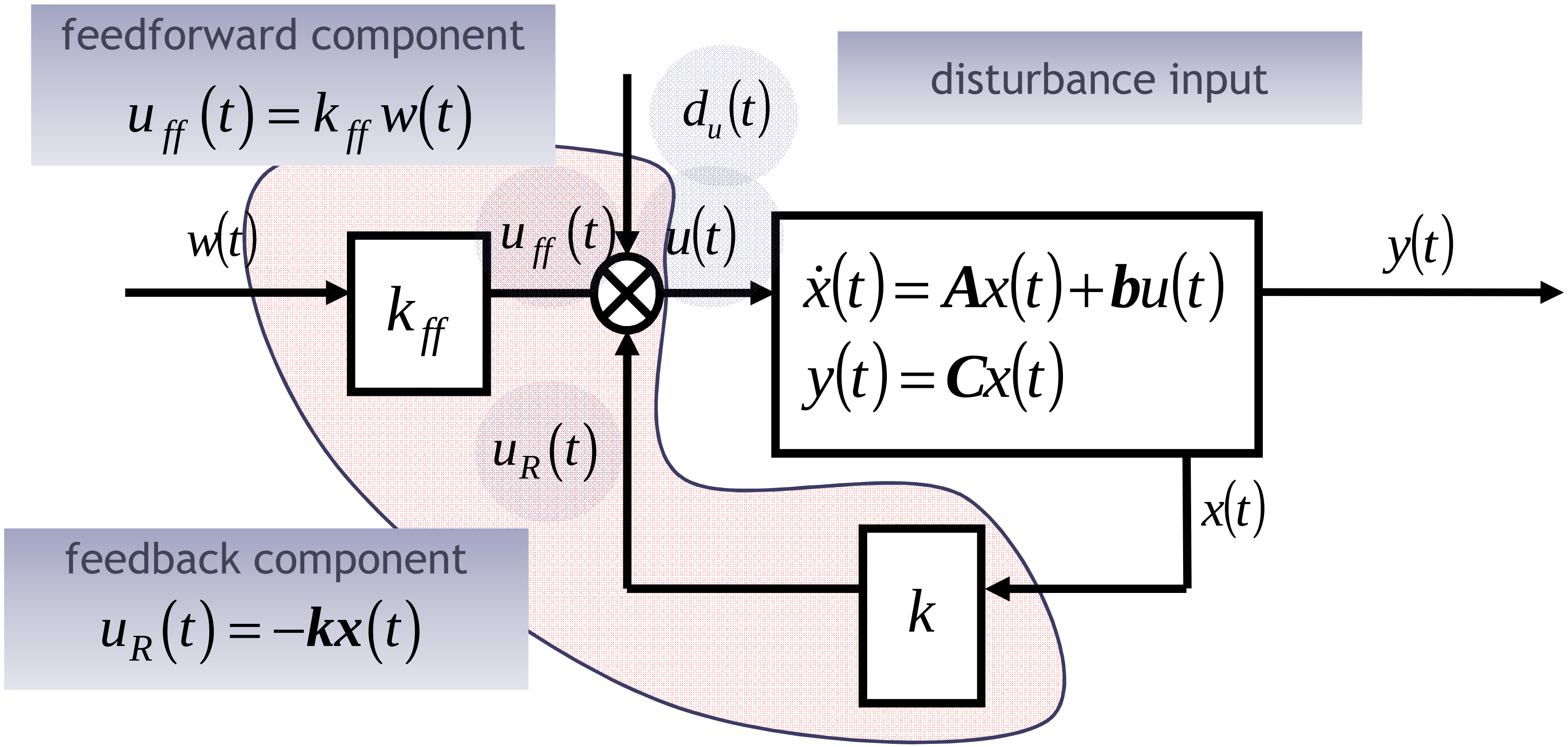


Inverted Pendula Control



B. 2)

State-feedback control - basic control scheme



$$u(t) = u_R(t) + u_{ff}(t) + d_u(t) = -\boldsymbol{kx}(t) + k_{ff} w(t) + d_u(t)$$

B. 2)

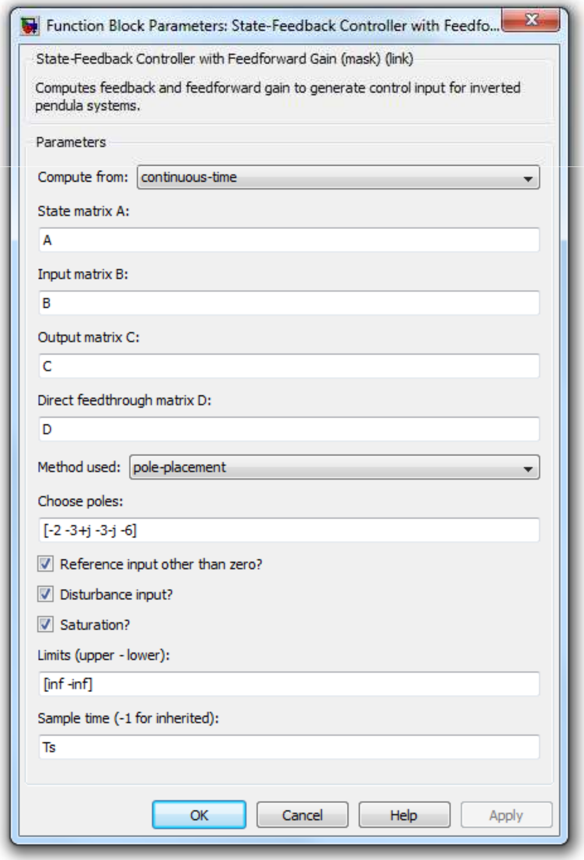
Library block *State-Feedback Controller with FeedForward Gain*

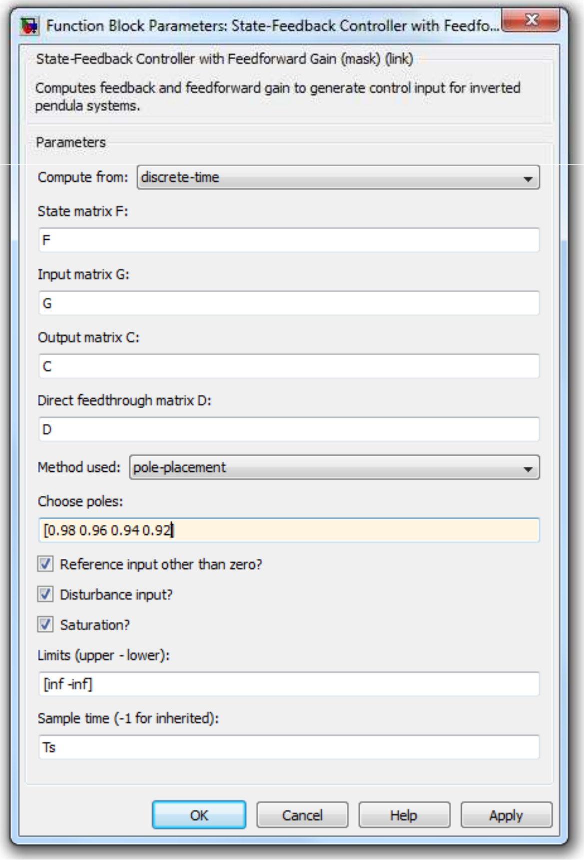
Computation of the feedback gain vector *k*

a) using the pole-placement algorithm

continuous-time state-space model

discrete-time state-space model



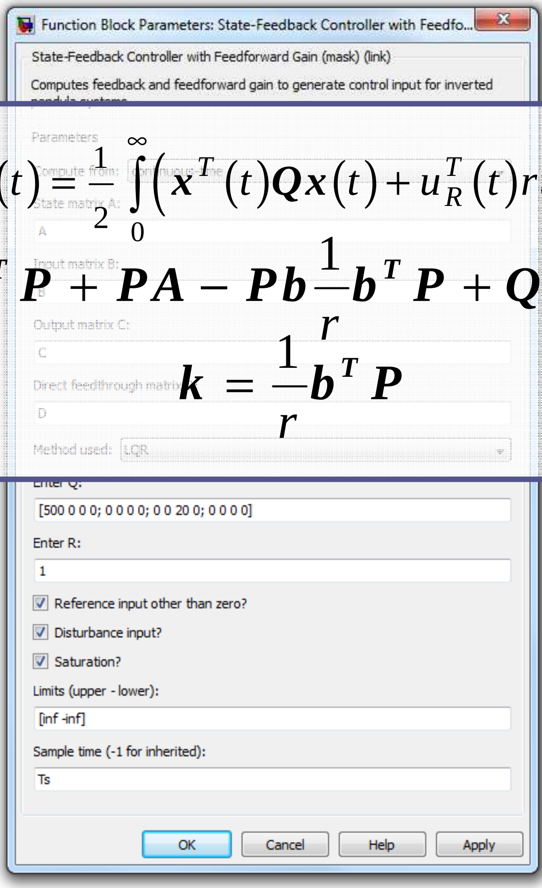


B. 2) Library block *State-Feedback Controller with FeedForward Gain*

Computation of the feedback gain vector *k*

2) using the linear quadratic optimal control method (LQR)

continuous-time state-space model

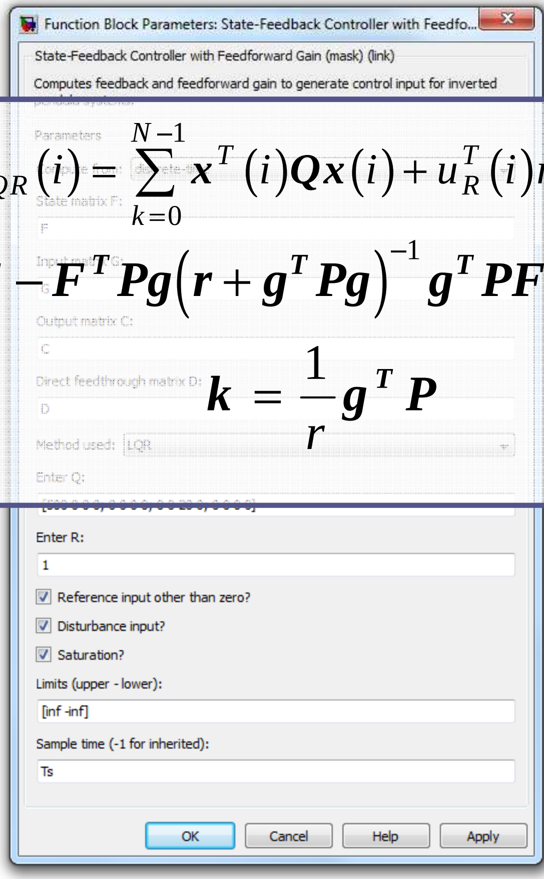


$$J_{LQR}(t)=\frac{1}{2}\int_0^{\infty}\left(\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t)+u_R^T(t)ru_R(t)\right)dt$$

$$\boldsymbol{A}^T\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A}-\boldsymbol{P}\boldsymbol{b}\frac{1}{r}\boldsymbol{b}^T\boldsymbol{P}+\boldsymbol{Q}=0$$

$$\boldsymbol{k}=\frac{1}{r}\boldsymbol{b}^T\boldsymbol{P}$$

discrete-time state-space model



$$J_{LQR}(i)=\sum_{k=0}^{N-1}\mathbf{x}^T(i)\mathbf{Q}\mathbf{x}(i)+u_R^T(i)ru_R(i)$$

$$\boldsymbol{F}^T\boldsymbol{P}\boldsymbol{F}-\boldsymbol{F}^T\boldsymbol{P}\boldsymbol{g}\left(\boldsymbol{r}+\boldsymbol{g}^T\boldsymbol{P}\boldsymbol{g}\right)^{-1}\boldsymbol{g}^T\boldsymbol{P}\boldsymbol{F}+\boldsymbol{Q}=0$$

$$\boldsymbol{k}=\frac{1}{r}\boldsymbol{g}^T\boldsymbol{P}$$

B. 2)

Library block *State-Feedback Controller with FeedForward Gain* - supported control objectives

- Initial deflection of the pendulum (nonzero initial conditions)
- Time-constrained and permanent disturbance input compensation ($d_u \neq 0$)
- Tracking a desired reference trajectory by the cart or arm

$$k_{ff} = \frac{-1}{c_1 \left(\boldsymbol{A} - \boldsymbol{bk} \right)^{-1} \boldsymbol{b}} \qquad k_{ffD} = \frac{1}{c_1 \left(\boldsymbol{I}_{2n+2} - \left(\boldsymbol{F} - \boldsymbol{gk} \right) \right)^{-1} \boldsymbol{g}}$$

State-Feedback Controller with Feedforward Gain

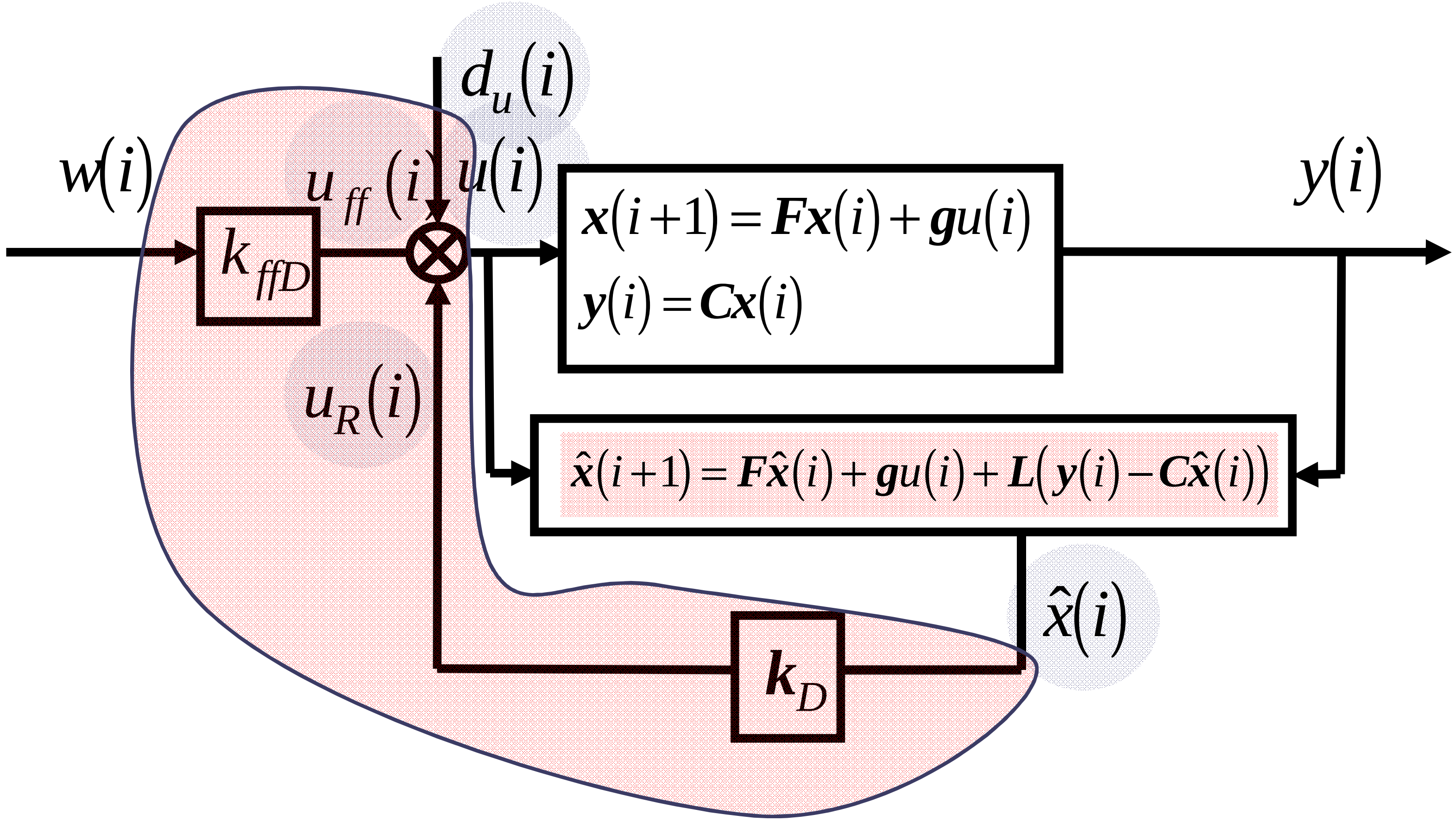
State-Feedback Controller with Feedforward Gain

State-Feedback Controller with Feedforward Gain

State-Feedback Controller with Feedforward Gain

B. 2)

State-feedback control with a discrete-time state estimator



$$u(i) = u_R(i) + u_{ff}(i) + d_u(i) = -\mathbf{k}_D\hat{\mathbf{x}}(i) + k_{ffD}w(i) + d_u(i)$$

B. 2)
Library block *Luenberger Estimator*

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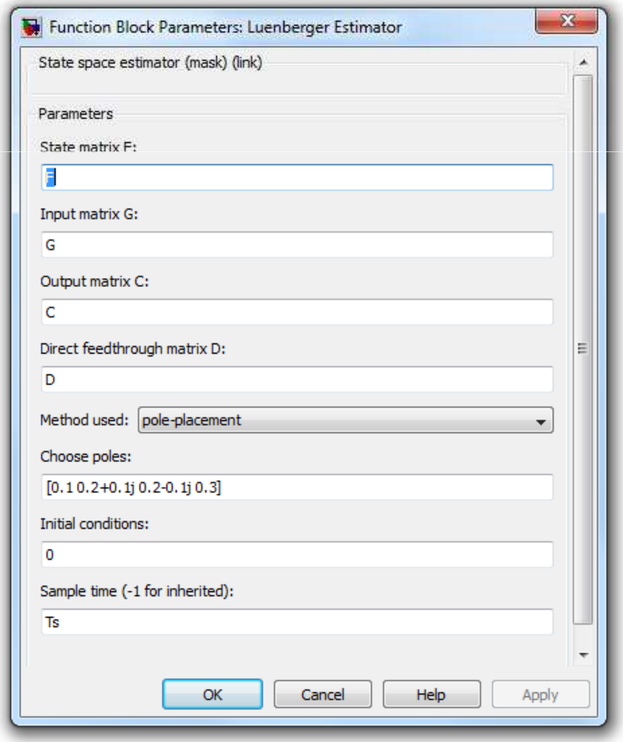
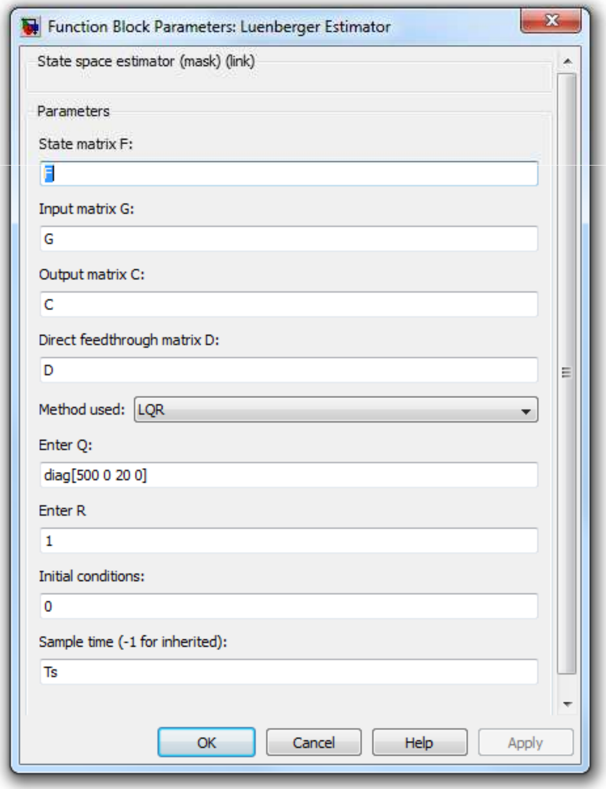
Computing the state estimator matrix ***L***

pole-placement algorithm



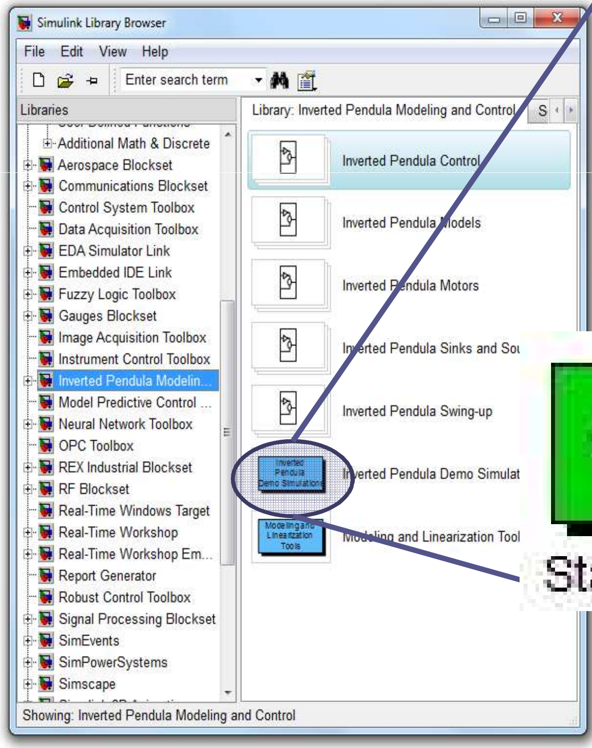
optimal control method (LQR)

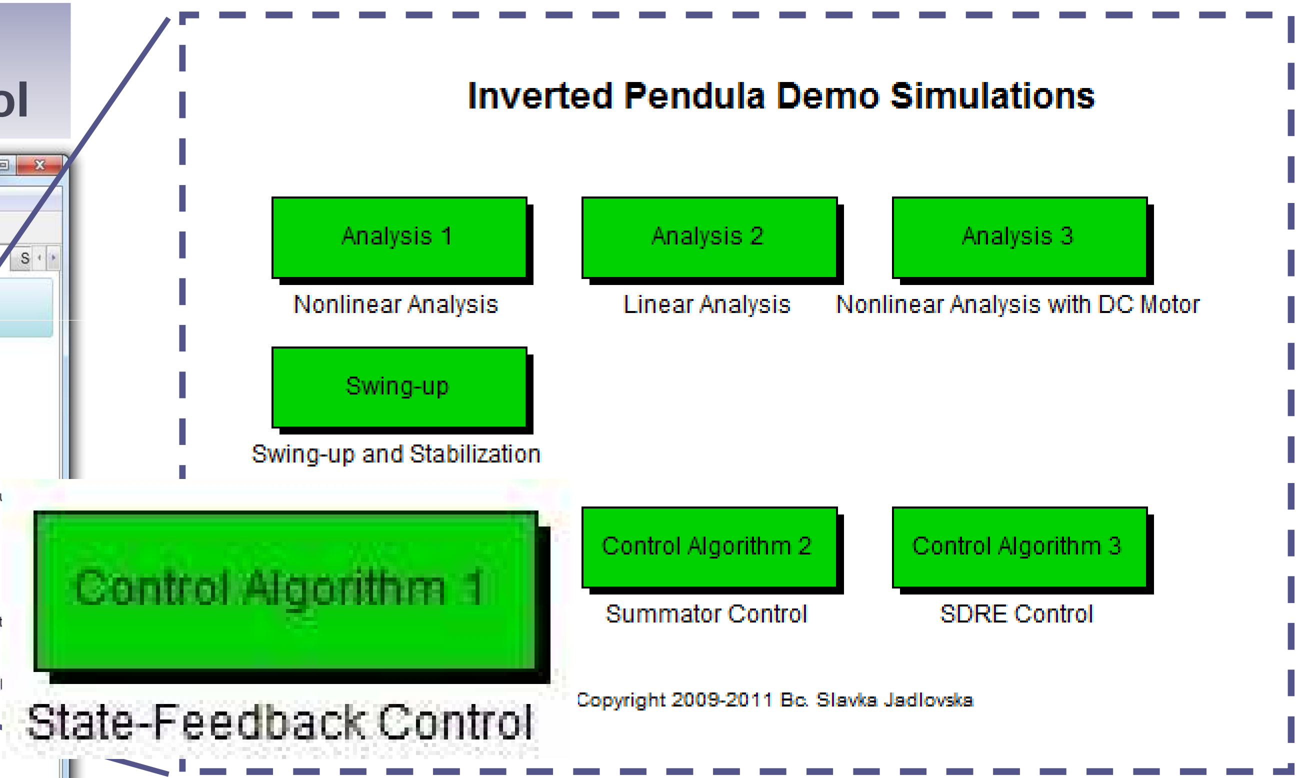


B. 2)

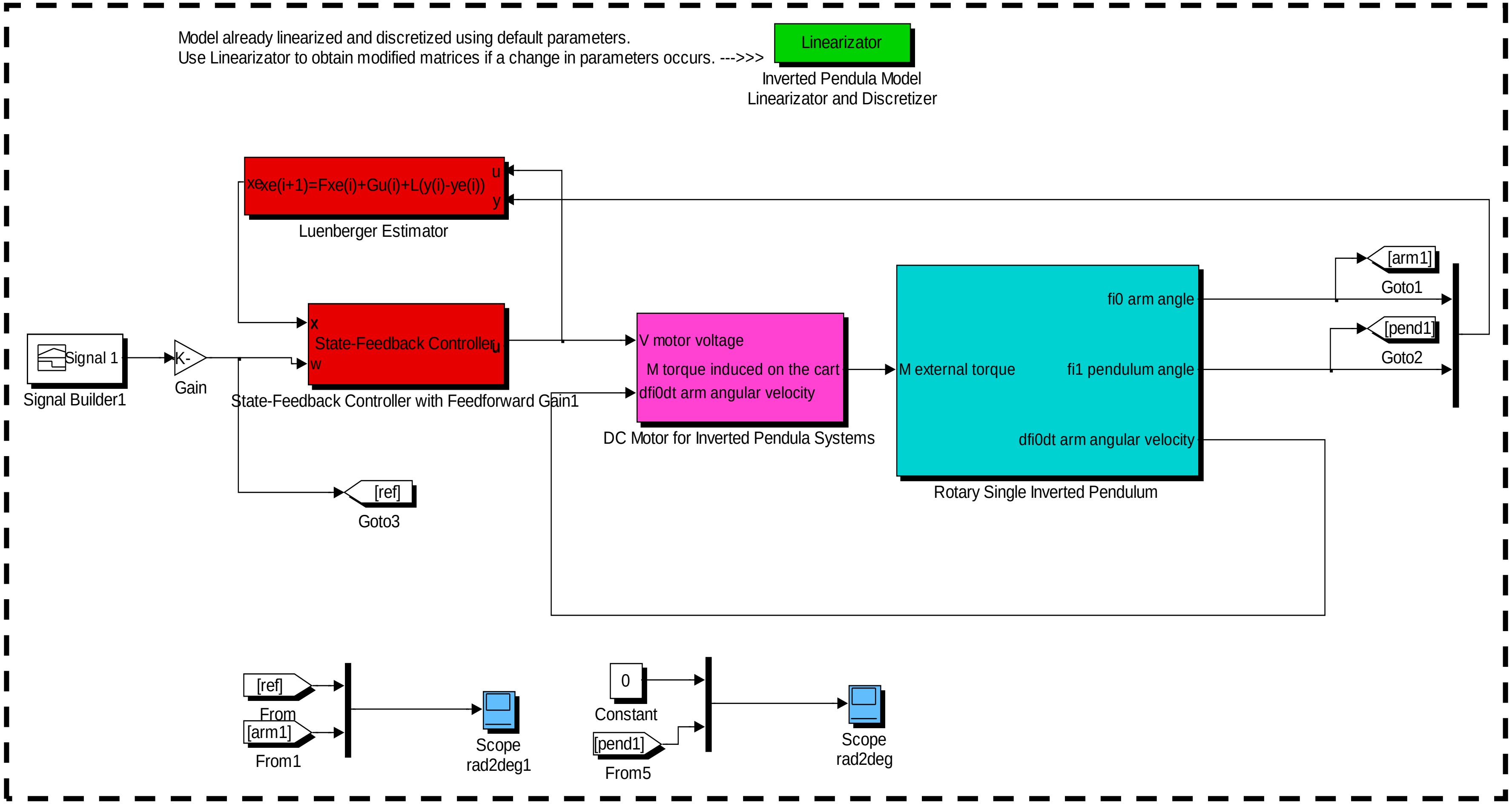
Demo Simulations III: State-feedback control for nonlinear force-torque / voltage models of inverted pendula systems

Inverted Pendula Modeling and Control



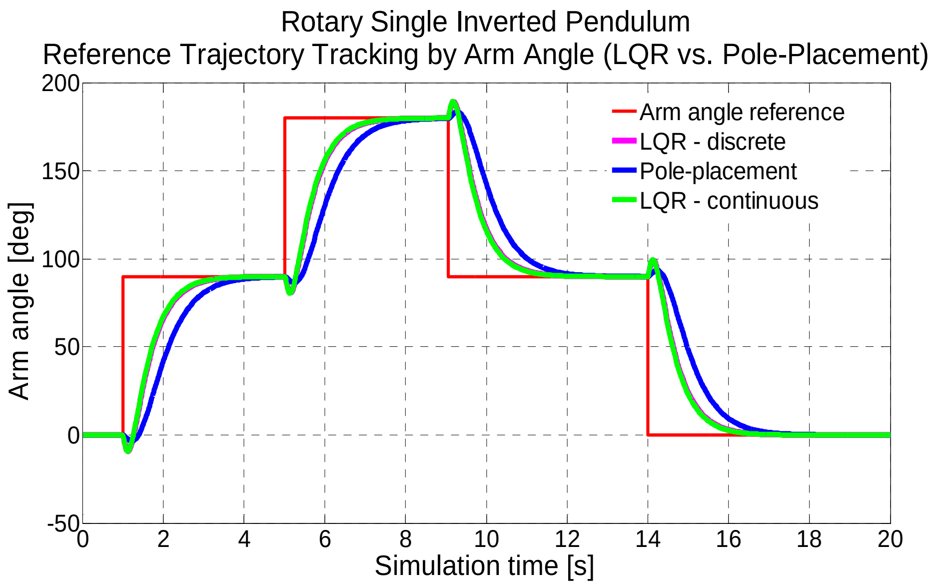


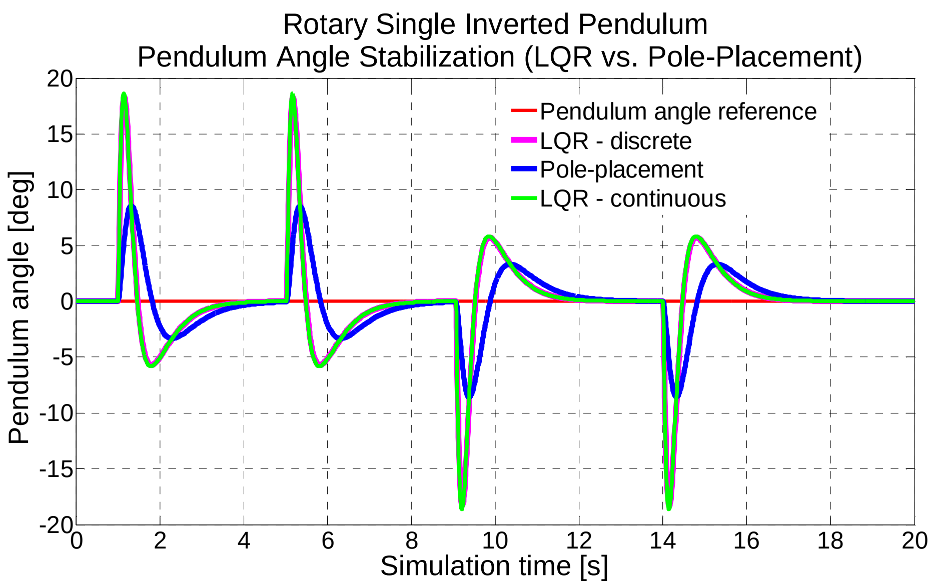
B. 2) Rotary single inverted pendulum (voltage model) - general simulation setup for state-feedback control



B. 2) Rotary single inverted pendulum - simulation results for LQR control compared to pole-placement

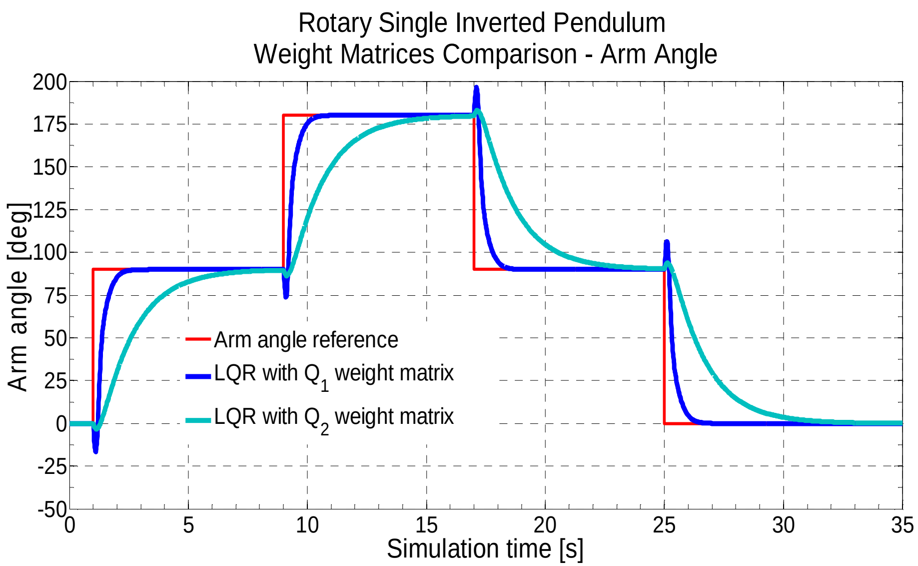
- control objective:
  - the arm should rotate for a total of half a circle and stop every quarter-turn to stabilize before returning to its initial position;
  - the pendulum should be kept upright all the time
- state-feedback control designed using continuous-time & discrete-time LQR and continuous-time pole-placement

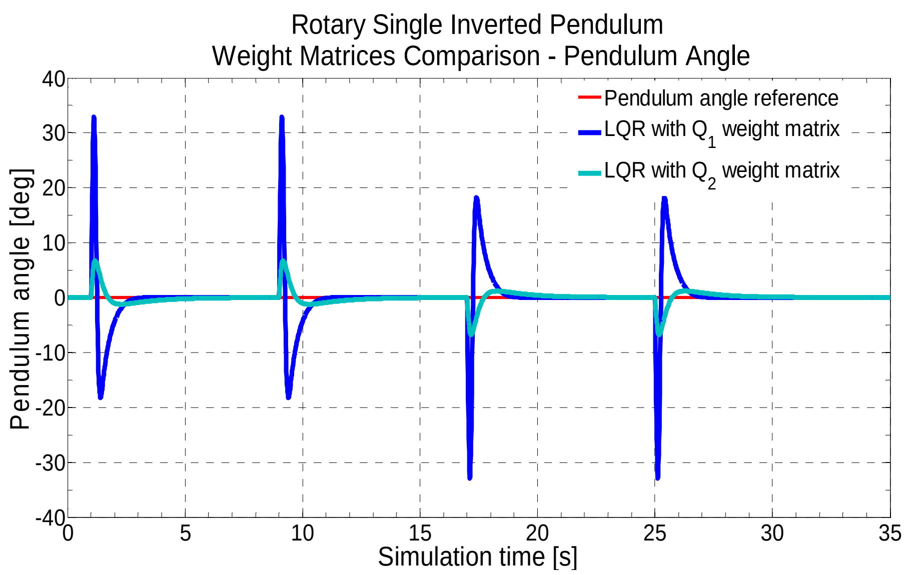




B. 2) Rotary single inverted pendulum - evaluation of the impact of weight matrices on system performance

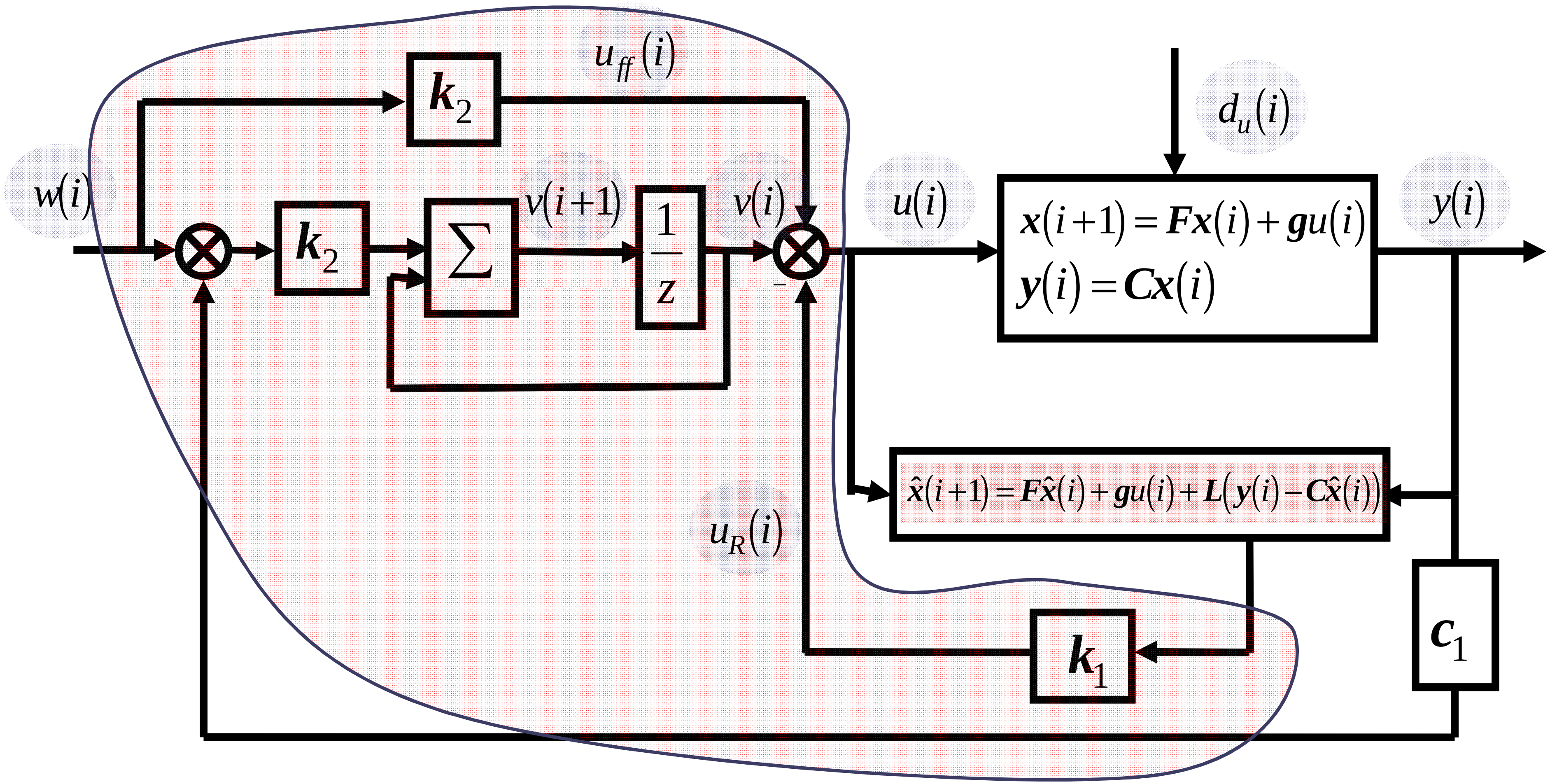
- tuning the functional weight matrices enables us to stress one of the two contradictory control objectives:
  - bringing the rotary arm into the desired position in the shortest time possible
  - stabilization of the pendulum in the upright position with the lowest possible overshoot



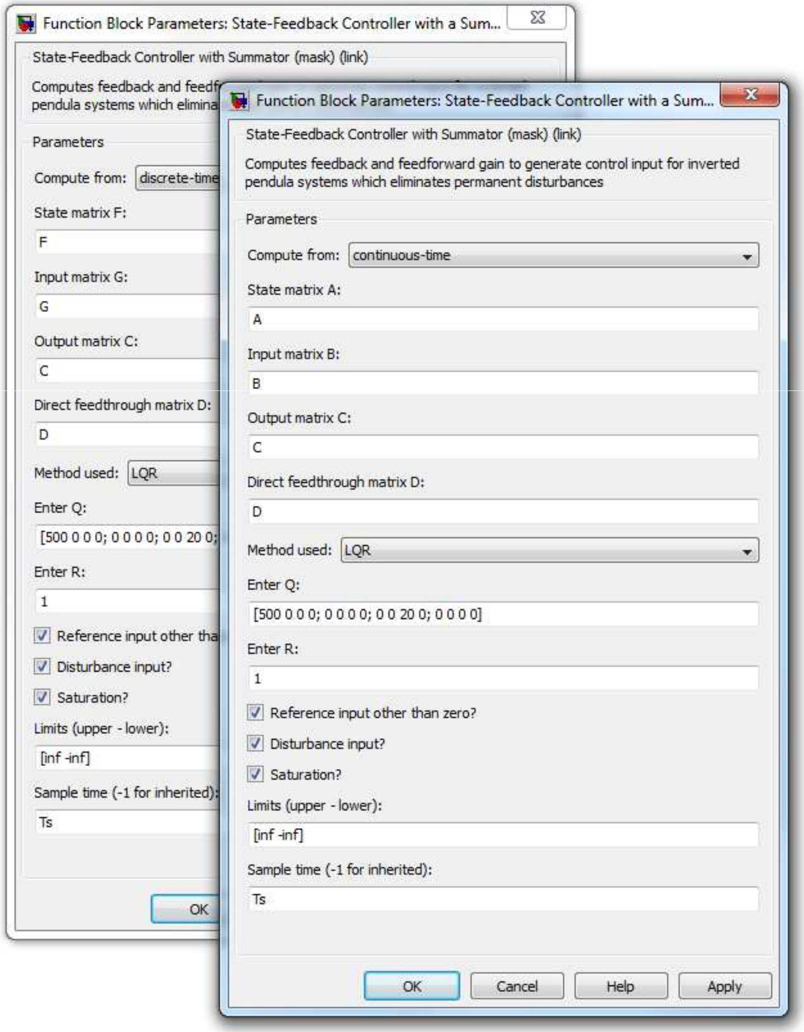


B. 2)

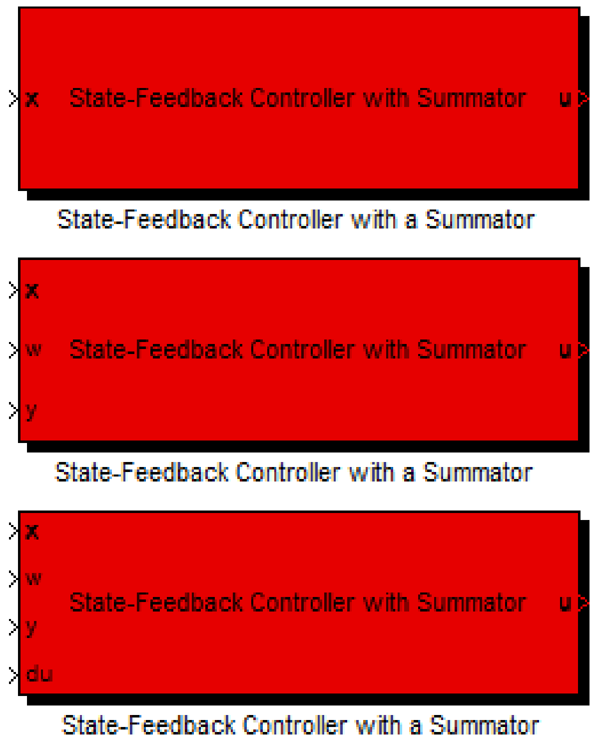
State-feedback control with permanent disturbance compensation using a summator - control scheme



B. 2)
Library block *State-Feedback Controller with a Summator*



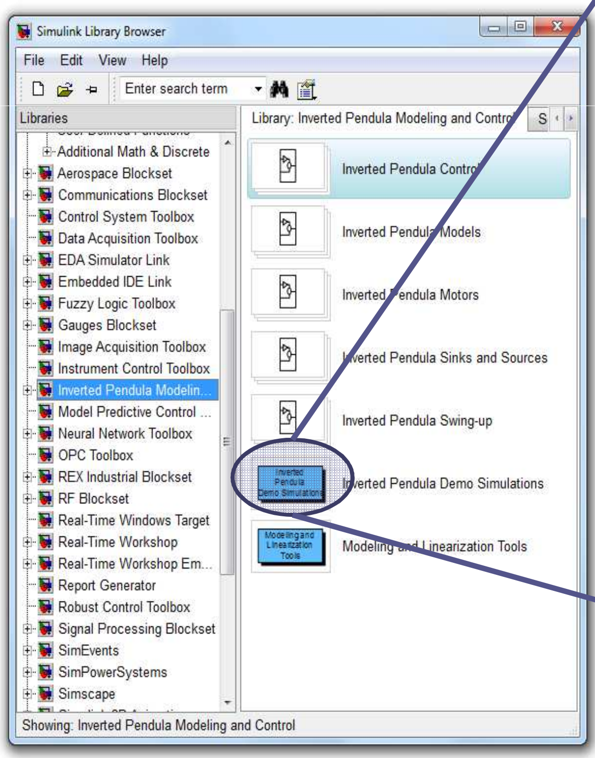
Control objectives:

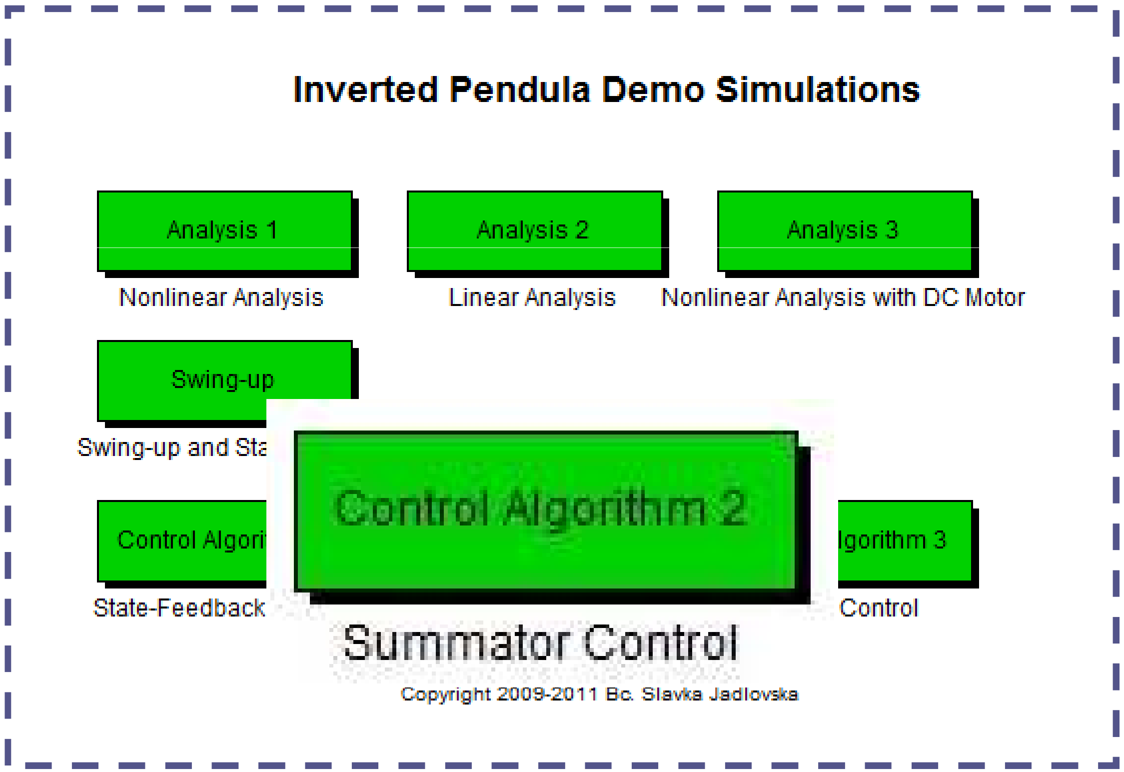


B. 2)

Demo Simulations IV: State-feedback control with a summator for nonlinear force-torque / voltage models of inverted pendula systems

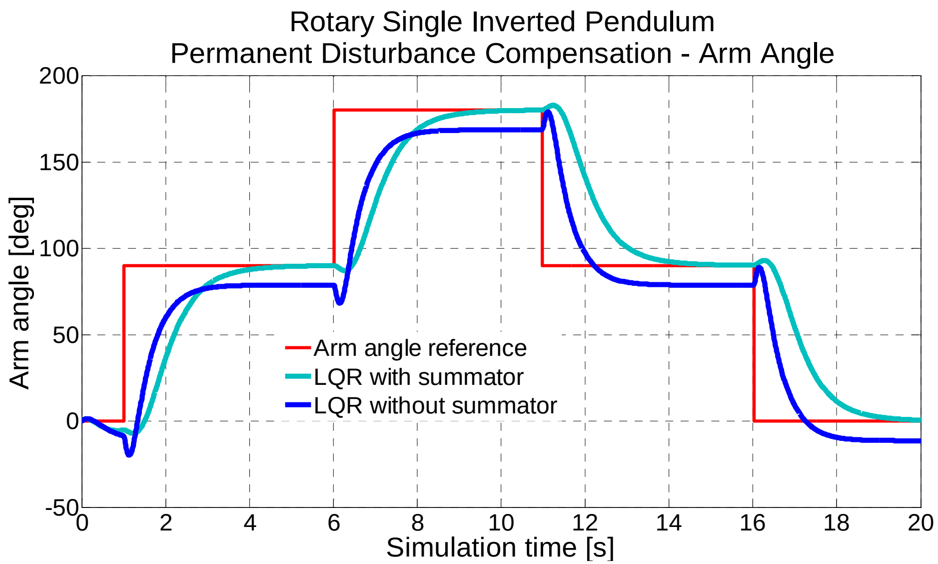
Inverted Pendula Modeling and Control

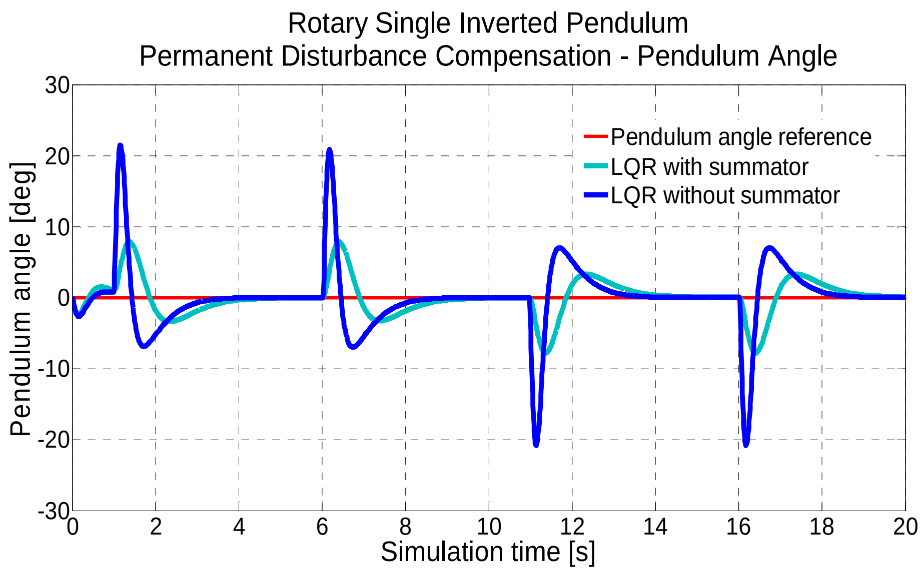




B. 2)
Rotary single inverted pendulum - simulation results for LQR control with a summator

- control objective: to track the reference trajectory & keep the pendulum upright with a constant disturbance input present
  - the **conventional LQR controller** fails to track the reference trajectory without producing steady-state error
  - permanent disturbances are successfully compensated by a **LQR algorithm with a summator included** in the control structure





Conclusion and Evaluation of Results

- comprehensive approach to modeling and control of the rotary single inverted pendulum system
- custom-designed Simulink block library ***Inverted Pendula Modeling and Control***
  - software framework for all covered issues (model derivation, open-loop analysis, linearization, state-feedback controller design)
  - provides suitable library blocks and original GUI applications to support every step of the process

Thank you for your attention.