Advanced Generalized Modeling of Classical Inverted Pendulum Systems

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Abstract

The purpose of this paper is to present the design and program implementation of expansions made to the existing general algorithmic procedure which yields the mathematical model for a classical inverted pendulum system with an arbitrary number of pendulum links. The expansions include the option to define the reference position of the pendulum in a planar coordinate system, to choose the reference direction of pendulum rotation and to select the shape of a weight attached to the last pendulum link. The underlying physical formulae based on the generalized inverted pendulum concept are implemented in form of a symbolic MATLAB function and a MATLAB GUI application. The validity and accuracy of motion equations generated by the application are demonstrated by evaluating the open-loop responses of simulation models of the classical single and double inverted pendulum system using newly-developed MATLAB blocks and applications.

Keywords: classical inverted pendulum system, attached weight, reference pendulum position, automatic model generation, symbolic MATLAB function

1 Introduction

Stabilization of a physical pendulum or a system of interconnected pendulum links in the upright unstable position is a benchmark problem in nonlinear control theory [1]: in recent years, several types of stabilizing mechanisms such as cart moving on a rail [2], rotary arm [3] or vertical oscillating base have been introduced. Inverted pendulum systems (IPSs) are therefore regularly employed as typical examples of *unstable nonlinear underactuated systems* in the process of verification of linear/nonlinear control strategies in corresponding control structures. Direct practical applications include walking humanoid robots, launching rockets, earthquake-struck buildings and two-wheel vehicles such as the *Segway PT*. Principles of modeling and control of IPSs can further be considered as the basic starting point for the research of advanced underactuated systems such as mobile robots and manipulators [4] as well as aircraft and watercraft vehicles [5].

We focused our research on the mutual analogy among mathematical models of IPSs with a varying number of pendulum links. Consequently, we introduced the concept of a *generalized n-link inverted pendulum system* with n+1 degrees of freedom (DOFs) and a single actuator, which allows to treat an arbitrary IPS as a particular instance of the system of n pendula attached to a given stabilizing base. General procedures which determine the Euler-Lagrange equations of motion for a user-specified instance of a generalized classical (i.e. on a cart) and rotary IPS were developed and implemented via MATLAB's *Symbolic Math Toolbox* [1].

The design and implementation of the general procedures has so far been based on the assumption of interconnected pendulum rods whose angle is determined relative to a fixed planar/spatial coordinate system and their center of gravity (*CoG*) is identical to their geometric center. The goal of this paper is to expand the existing procedures for automatic model generation of classical IPSs with further, practically motivated generalizations. The paper is organized as follows. Firstly, the generalized classical IPS which has been studied so far is presented, this time as the basic starting point for further research. Two categories of expansions are next described in terms of their impact on general mathematical model derivation: change in *CoG* position of a pendulum caused by a weight attached to its end and the option to specify the orientation of the pendulum reference position, together with the reference direction of pendulum rotation. The next section details the implementation of an expanded general procedure whose aim is to ultimately cover all possible forms of models found in relevant literature by including all possible combinations of underlying assumptions for pendulum reference position/direction and the existence of attached weights. Finally, the validity and accuracy of the procedure is verified using the classical single and double IPS, both represented by pre-prepared Simulink blocks encompassing all investigated features, and the paper is concluded with an evaluation of achieved results.

2 Expanded Generalized Classical Inverted Pendulum System

The *generalized system of classical inverted pendula* was introduced in [1] as a set of $n \geq 1$ rigid, homogenous, isotropic rods (*pendulum links*) which are interconnected in joints and attached to a stable cart which enables movement along a single axis. A multi-body mechanical system defined this way is *underactuated* since it has fewer actuators than DOFs [4]: the only input (force $F(t)$ acting upon the cart) actuates the cart position [m] as well as the n pendulum angles [rad]. Through the mathematical model derivation process in [1], it was assumed that all motion was bound to a planar coordinate system with the cart moving along the x-axis, which was simultaneously identified with the projection of the zero potential energy level into the *xy*-plane. The value of every pendulum angle was determined clockwise with respect to the vertical upright position of the pendulum, which was defined as parallel to the *y*-axis (Fig. 1).

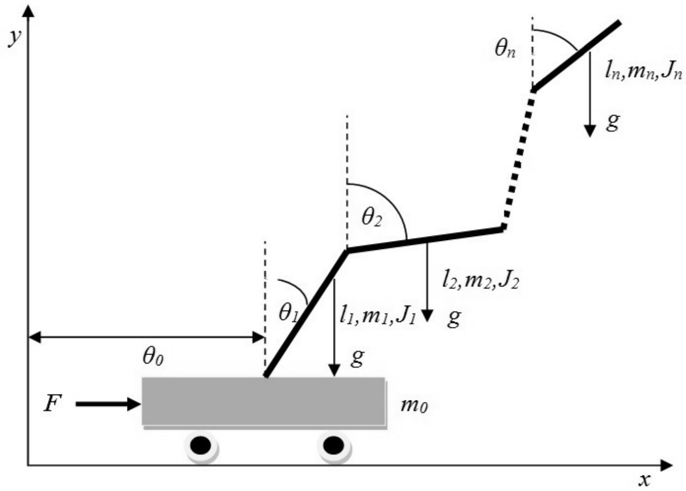

Fig. 1. Generalized system of classical inverted pendula – scheme and basic nomenclature

According to the Lagrangian formulation of classical mechanics, every possible configuration of a multi-body system can be uniquely defined by a vector of generalized coordinates equivalent to the system's DoFs which are, in the case of a generalized classical IPS, identified as cart position and pendulum angles:

$$\boldsymbol{\theta}(t) = \left(\theta_0(t) \quad \theta_1(t) \quad \dots \quad \theta_n(t)\right)^T \tag{1}$$

For every generalized coordinate, a nonlinear second-order differential motion equation is specified by employing *Euler-Lagrange equations* defined in the form:

$$\frac{d}{dt}\left(\frac{\partial L(t)}{\partial \dot{\boldsymbol{\theta}}(t)}\right) - \frac{\partial L(t)}{\partial \boldsymbol{\theta}(t)} + \frac{\partial D(t)}{\partial \dot{\boldsymbol{\theta}}(t)} = \boldsymbol{Q}^*(t) \tag{2}$$

where $L(t)$ is the difference between the multi-body system's kinetic and potential energy, $D(t)$ stands for the dissipation properties and $\boldsymbol{Q^*}(t)$ is the vector of *generalized external inputs* [6]. The process of mathematical model derivation for a selected IPS hence transforms into a procedure to determine its kinetic, potential and dissipation energy, each defined as a sum of energies of the multi-body system's individual bodies, i.e. the cart (i=0) and all pendulum links (i=1,... n):

$$E_K(t) = \sum_{i=0}^{n} E_{Ki}(t) \;, E_P(t) = \sum_{i=0}^{n} E_{Pi}(t) ,\; D(t) = \sum_{i=0}^{n} D_i(t) \tag{3}$$

While the potential energy of i-th body depends on *CoG* position coordinates, *CoG* velocity components need to be obtained to specify kinetic energies or dissipative properties [7]. The actual physical formulae which form the core of the procedure for a generalized IPS were derived in [1] and presented together with generated motion equations of example IPSs and the verification of their validity.

2.1 Modified Mass Distribution as a Result of Attached Weight

As the first expansion to the generalized IPS, we considered a system with an arbitrary number of inverted pendulum links mounted on a cart, in which a weight with a specified shape is firmly attached to the end of the pendulum link furthermost from the cart. Therefore, for i=1,..., n-1, the whole mass of a pendulum rod is concentrated to the *CoG* located midway from the pivot point, but for $i=n$, the attached weight causes the *CoG* of a *pendulum link with weight* to shift away from the geometric center of the homogenous rod.

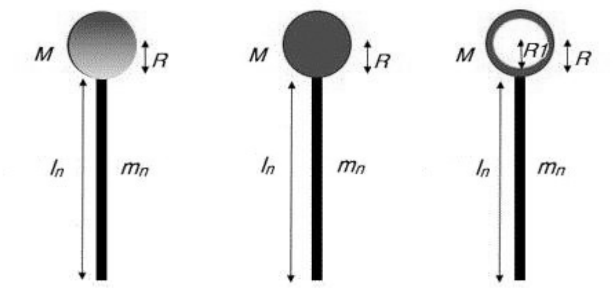

Fig. 2. Considered weight shapes: sphere, cylinder, ring

By computing the distance between the *CoG* of the weighted pendulum link and the pivot point, *CoG* position and velocity coordinates followed by related potential and kinetic energies were derived and can be found in [8]. To fully express the kinetic energy, the moment of inertia of the weighted pendulum was calculated as the sum of moments of inertia of the pendulum rod and the weight itself. Three shapes of weight – sphere, cylinder and ring (Fig. 2) were considered, however the algorithm can be applied to all symmetric isotropic bodies with known moments of inertia, since the resulting equations of motion only differ by the moment of inertia of the attached weight.

2.2 Modified Pendulum Reference Position

Most differences between the correctly derived inverted pendulum models found in various sources can be attributed to the initial choice of reference position for all pendulum links and the reference direction of pendulum rotation, both of which determine the numeric value of the pendulum angle at every time instant. Our next goal was therefore to expand the procedure of mathematical model derivation for generalized IPS so that all feasible combinations of initial assumptions would be covered. Eight possible combinations of reference pendulum positions with respect to a planar coordinate system (*top*, *bottom*, *right*, *left*) and pendulum movement directions (*clockwise*, c*ounterclockwise*) were considered (see Fig. 3 for examples). During the derivation process, which was recorded step-by-step in [9], the selected combination of initial assumptions was shown to have direct influence on the coordinates of *CoG* position of each pendulum link and subsequently on the related *CoG* velocity components, on the expressions for kinetic and potential energy, and finally, on the motion equations in their final form.

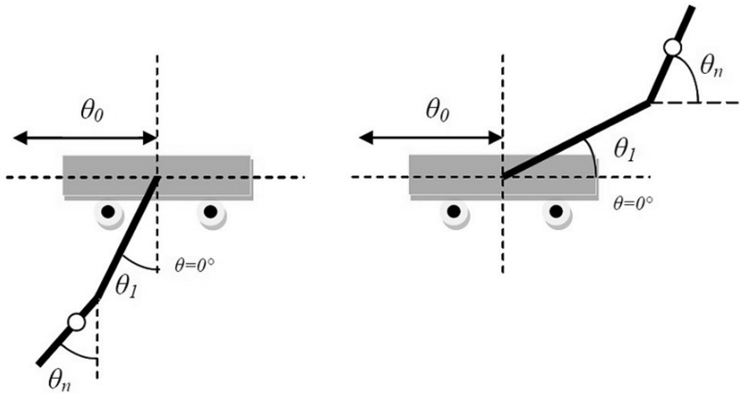

Fig. 3. Examples of reference frame definitions – a) bottom clockwise b) right counterclockwise

3 Expanded General Procedure for Mathematical Model Derivation – Program Implementation and Application

Using the familiar theoretical background as well as the newly-derived physical formulae, we completely reworked the earlier general algorithmic procedure so that it would yield a mathematical model of an arbitrary classical IPS with respect to user-selected criteria for attached weights and pendulum reference position. The procedure was once again implemented as a MATLAB function which generates the nonlinear equations of motion via *Symbolic Math Toolbox* in the simplified and rearranged form, equivalent to the most likely form obtained by manual derivation. An application with graphical user interface, *Inverted Pendula Model Equation Derivator_v2*, was also developed to provide a user-friendly access to the function. Compared to the earlier version of the *Derivator* where the user could only select the number of pendulum links [10], four options for weight type (including none) and eight for reference position/direction are now provided. As a further improvement, the equations can now be displayed in form of LaTeX expressions in addition to the original MATLAB representation. Fig. 4 shows an example preview of the *Derivator_v2* window which contains the generated model equations for the classical double IPS with an attached cylinder-shaped weight.

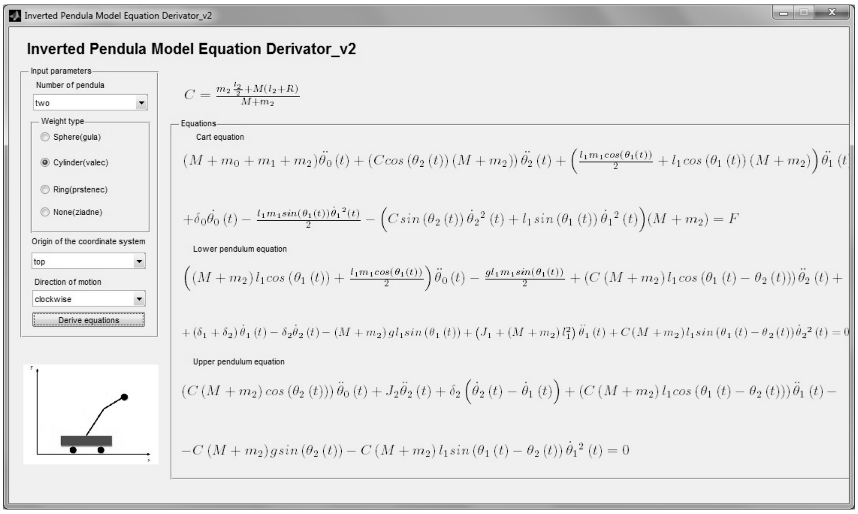

Fig. 4. *Inverted Pendula Model Equation Derivator – ver. 2* – GUI application for mathematical model generation for classical IPSs, „C“ is the *CoG*-to-pivot distance for the *n*-th pendulum link

By evaluating the results of the implemented expanded general procedure, we concluded that the mathematical model of the original generalized classical IPS (i.e. without attached weight) represents a special case of a model of the weighted IPS in case the weight mass is set to zero, which serves as a confirmation of the procedure’s accuracy. Analogically, the original specification of a generalized IPS, characterized by a particular combination of initial assumptions about the reference position and rotation of the pendulum, now becomes a representative of a family of models related to the same physical system with confirmed validity.

3.1 Verification of Generated Mathematical Models of Classical Inverted Pendulum Systems

A structured *Simulink* block library, *Inverted Pendula Modeling and Control* (*IPMaC*), has been developed since 2009 as a comprehensive software framework for the analysis and control of IPS in the simulation environment. To reflect the expansions outlined in previous sections, library blocks which implement the motion equations of the *classical single* and *classical double* IPS were equipped with newly-implemented properties. As a result, the subsystem mask of both blocks now allows the user to dynamically change the parameters of the simulation model, specify the number of input and output ports, the shape of attached weight, reference pendulum angle value and reference direction of pendulum rotation. The possibility to switch to a simplified model, which neglects friction and omits the backward impact of the pendulum links on the cart, was also implemented. The necessity to create a separate block for each set of equations was eliminated by callbacks which ensure the dynamic adjustment of the block structure, so that it will always correspond to a specific set of motion equations.A GUI application, *Analysis of Inverted Pendulum Systems* (*AoIPS*) was developed in MATLAB as a graphical tool to monitor, analyze and evaluate the open-loop dynamics of a selected classical IPS in a single window. For every simulation experiment, a scheme containing a suitable Simulink block is run in the background, block parameters are set to values specified in the GUI, and simulation results are exported into separate figures for further investigation.

It will next be assessed whether the generated mathematical models of classical single and double IPSs can be considered as valid and accurate for control design purposes. Using the *AoIPS* tool, open-loop responses to an impulse signal constrained in time/amplitude were obtained for both simulated models, starting from the initial upright equilibrium. Numeric parameters were specified in [8][9].

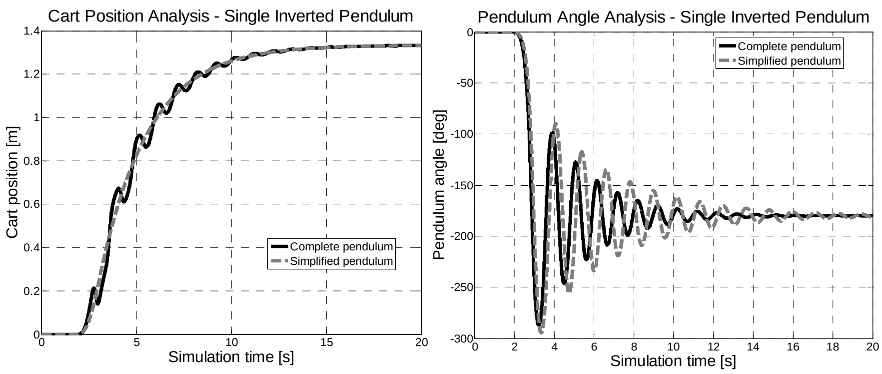

Fig. 5. Classical single inverted pendulum system – cart position and pendulum angle in a complete vs. simplified model

Simulation experiments which illustrate the dynamics of simplified models of IPSs or analyze the influence of the pendulum's modified mass distribution on the system dynamics were evaluated first. The comparison of time behavior of the cart position and pendulum angle for the complete and simplified model of a *classical single* IPS is depicted in Fig. 5, while the dynamics of the *classical double* IPS with different attached weights is evaluated in Fig. 6. In both cases, all pendulum links fall from the upright into the downward equilibrium through a damped oscillatory transient state and stabilize there, in compliance with the empirical observations of pendula behavior. In the simplified model, the pendulum makes a very long transition to the steady state as a result of neglected friction, and the cart trajectory correctly shows no signs of the backward impact caused by the pendulum movement. If weight is attached to the upper pendulum, the pronounced "jerky" cart movement is caused by the inertia of the heavier pendulum link. Total damping of a weighted system is much lower than that of the system with no weight load, which is reflected on larger oscillations of pendulum links and their prolonged settling time. The differences between the dynamics of systems with different types of weights are minimal.

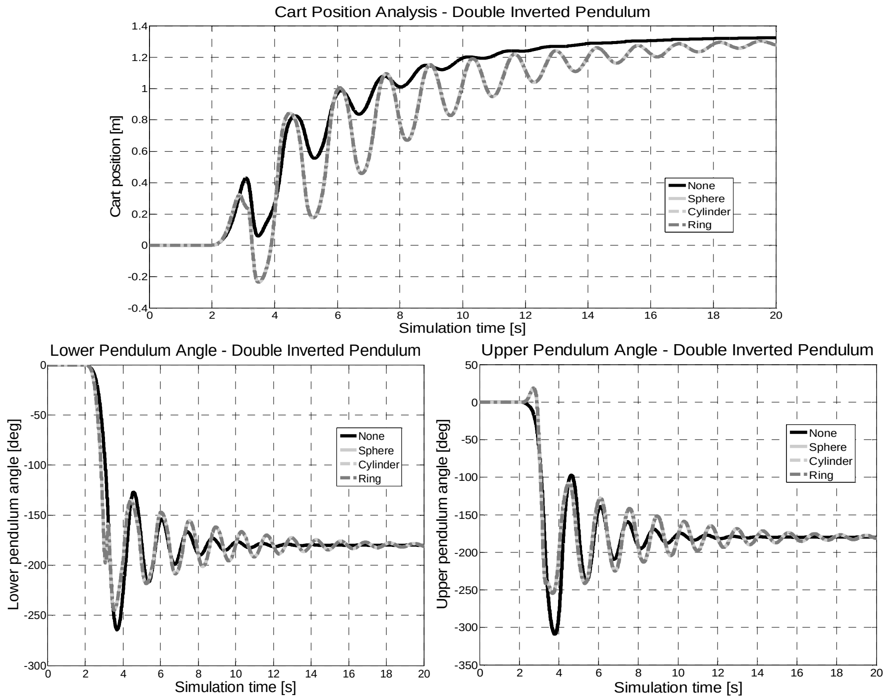

Fig. 6. Classical double inverted pendulum – cart position and pendula angles for different shapes of weights, including no weight

The following simulation experiments evaluate the influence of combinations of initial assumptions on the response of IPSs. Fig. 7 depicts the dynamical behavior of a *classical single* IPS after selecting four starting positions which vary by 90° (*top, left, down, right*), while the pendulum angle is determined clockwise in all cases. The effect of the direction in which the pendulum angle is determined (*clockwise / counterclockwise*) is shown on a *classical double* IPS in Fig. 8. It has been proven that the changes in initial assumptions have no effect on the dynamics of either the cart or the pendulum links, and only the graphical representation of pendulum behavior is subject to change. Choice of the reference value of the pendulum angle determines the numeric value corresponding to the upright/downward position of the pendulum, and the selected reference direction defines whether the pendulum angle will increase or decrease during simulation, as it is clear from the „mirror image“ depicting the pendula behavior in Fig. 8.

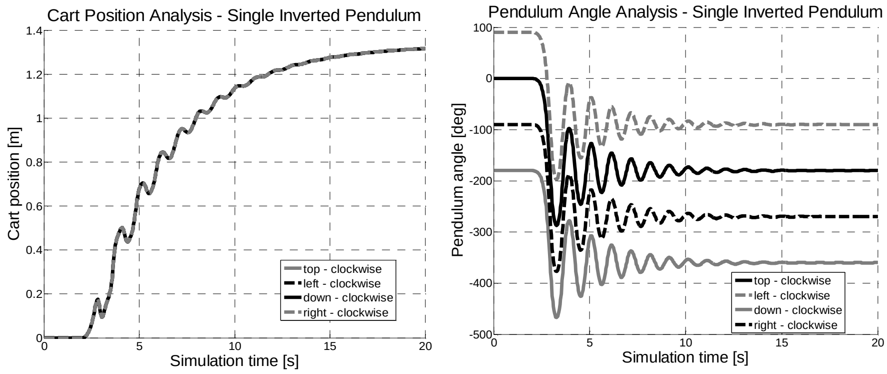

Fig. 7. Classical single inverted pendulum system – cart position and pendulum angle – effect of the changing reference position of the pendulum – top, left, down, right

Reasonable behavior of the open-loop responses of both simulation models means that under all criteria, systems described by the generated motion equations can be considered accurate enough to serve as a reliable testbed for the verification of linear and nonlinear control algorithms.

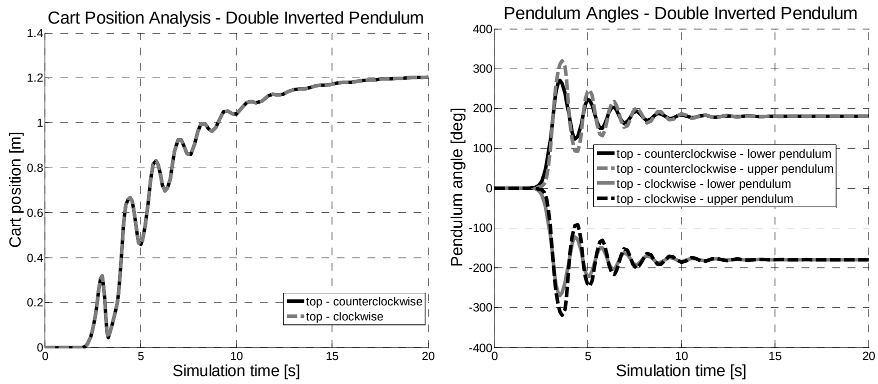

Fig. 8. Classical double inverted pendulum system – cart position and pendulum angles determined in a clockwise / counterclockwise reference direction

Conclusion

The purpose of this paper was to expand and further generalize the existing algorithmic procedure for obtaining the equations of motion of classical inverted pendulum systems (IPSs) with an arbitrary number of pendulum links. The expanded general procedure covers all feasible combinations of initial assumptions for the pendulum reference position and direction of rotation, and considers various shapes of weight load attached to the last pendulum link. A GUI application was developed to provide an intuitive interface to the MATLAB function which implements the procedure. The validity of generated motion equations was confirmed by evaluating open-loop responses of simulation models of classical single and double IPS with emphasis on the newly-introduced features.

The results of this paper allow the control engineer to effortlessly obtain a highly accurate, error-free mathematical model of a selected IPS, simplifying the process of model-based control design. Moreover, the readily available collection of mathematical and simulation models of IPSs can be regarded a testbed model basis for exploring properties of underactuated mechanical systems and consequently, as a starting point for research in mobile and manipulator robotics.

Acknowledgments

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