Solving Economical Problems by Vector Optimization Methods

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***Abstract* — This paper deals with solving economical problems by methods of vector optimization. Description and solving vector optimization tasks using chosen methods is the main issue. Paper also contents example of economical vector optimization problem, which can be specified as problem of choosing the ideal investment strategy. Application in MATLAB was created for intuitive and simple solving vector optimization tasks.**

***Keywords* — economical problems, linear programming, quadratic programming, vector optimization**

I. Introduction

Optimization is defined as choosing the best option from the set of possible options. Optimization tasks and problems take place in many different sectors. The main goal in economical sector is to maximize profit or minimize costs. It can be used also in solving traffic problems, specifically in minimizing time and costs of transportation. Also consumers might use optimization methods to satisfy their needs as much as possible with minimum costs. Publications dealing with optimization and vector optimization task are [1] and [4]. Main asset of this work, based on [8], is creating MATLAB application for simple and intuitive solving vector optimization tasks, because there wasn't any complex application able to solve these tasks. Application provides users with user interface for solving vector optimization problems with two objective functions. Using of application is illustrated by solving problem of defining the optimal investment strategy using vector optimization methods.

II. Brief overview of optimization methods

Optimization process is disposing with apparatus to choose the best possible solution for particular task. In order to make final decision responsibly and competently, it is necessary to:

1. create mathematical model which precisely describes the situation. Model represents mathematical description of real system. It has to contain quantifiable parameters , which are used to evaluate the success rate of optimization process (profit, costs etc.).Model can also contents some constrains (f.e. maximal amount of invested money);
2. find the solution of particular optimization problem using proper algorithm, because there is no universal algorithm, which can be used to solve all optimization tasks (tasks solved by methods of linear and quadratic programming are described in this paper);
3. verify and analyze found solution (if it is the real solution of optimization task). It is also necessary to interpret the solution correctly.[1]

General optimization problem can be defined as minimization (maximization) of objective function

$$f = \{x_1, x_2, \dots, x_n\} \tag{1}$$

with respecting all constrains

$$g_i = \{x_1, x_2, \dots, x_n\}, \qquad for\ i = 1,2,\dots,n$$
$$x_j > 0, \qquad for\ j = 1,2,\dots,n. \tag{2}$$

Methods of mathematical programming are used to solve optimization tasks. Depending on the type of objective function, these methods can be divided into:

- linear programming methods,
- non linear programming methods,
- integer programming methods
- parameter programming methods,
- stochastic programming methods.

In following parts of the paper there will be a more detail description of linear and quadratic programming methods used to solve optimization tasks. Values of these solutions will be subsequently used in vector optimization process.

A. Linear programming task description

Linear programming is part of mathematical programming dealing with finding optimal solutions of optimization tasks.

Most of optimization tasks consists of two parts:

- main goal – defined by objective function,
- constrains .

Both of these parts are described by linear functions with multiple variables. Generally the linear programming problem can be described with following mathematical model:

$$z = c_1x_1 + c_2x_2 + \dots + c_nx_n \ \rightarrow opt(min, max) \tag{3}$$

with respecting constrains

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n >=< b_1,$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n >=< b_2,$$
$$\dots \dots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n >=< b_n. \tag{4}$$

Other way to define the task is to use the matrix form:

$$z = c^Tx \rightarrow opt(min, max)$$
$$Ax >=< b, \tag{5}$$

in which *c* is vertical vector of objective function coefficients, *x* is vertical vector of task variables , *A* is matrix describing left sides of constrains a *b* is vertical vector describing right sides of constrains.[1]

B. Quadratic programming task description

Goal of quadratic programming task is to optimize quadratic objective function with linear constrains. To describe general form of quadratic programming task, there is a need to mention some terms inevitable for closer definition of this form.

There is a symmetrical and positively semidefinite matrix $C \in \mathbb{R}^n$ and there is d ,$a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, $i = 1,2,\dots,m$. Let's label:

$$P = \{\, x \ \in \mathbb{R}^n, x_1, \dots, x_s \geq 0, s \ \in \{0, \dots, n\}\}$$
$$X = \{\, x \ \in P\langle a_i, x\rangle \leq b_i, i = 1, \dots, k; \langle a_i, x\rangle = b_i, i = k+1, \dots, m\} \tag{6}$$

After this definition we can define general form of quadratic programming task as

$$f(x) = \frac{1}{2}\langle Cx, x\rangle + \langle d, x\rangle \rightarrow min, x \in X. \tag{7}$$

We will label $\mathrm{A} = [\mathrm{a}_1, \dots, \mathrm{a}_\mathrm{m}]^\mathrm{T}$, where $\mathrm{a}_1, \dots, \mathrm{a}_\mathrm{n} \in \mathbb{R}^\mathrm{n}$ are rows of matrix A a $\mathrm{b} = [\mathrm{b}_1, \dots, \mathrm{b}_\mathrm{m}]^\mathrm{T}$. With closer knowledge it is possible to define Lagrange function of quadratic programming task:

$$L = \frac{1}{2}\langle Cx, x\rangle + \langle A^Ty + d, x\rangle - \langle y, b\rangle, \tag{8}$$

and so

$$L_x^{'}(x, y) = Cx + A^Ty + d. \tag{9}$$

It is important to mention, that $\mathrm{y} \ \in \mathrm{Q} = \{\mathrm{y} = (\mathrm{y}_1, \dots, \mathrm{y}_\mathrm{m}), \mathrm{y}_1, \dots \mathrm{y}_\mathrm{k} \geq 0\}$.

Solution of quadratic programming task is x^* just when there is $\mathrm{y}^* \in \mathrm{Q}$ respecting so called Kuhn – Tucker conditions:

$$\langle Cx^* + A^Ty^* + d, x - x^*\rangle \geq 0, \forall x \in P$$
$$y_i^*(Ax^* - b)_i = 0, i = 1, \dots, m. [3] \tag{10}$$

III. VECTOR OPTIMIZATION DESCRIPTION AND METHODS

Vector optimization is dealing with ways of optimizing the problem with multiple goals. It is used when it is necessary to accept more than one factor in finding the ideal solution of optimization task. This type of optimization was created for solving planning and organizing problems in manufacturing process. Nowadays it is used in many different areas (f.e.in dynamic management systems).

Vector optimization task is defined with relation to controlled system. It is described with *n*-dimensional vector $(x_1, x_2, \ldots, x_n)$, with $x \in \{X\}$ and is evaluated with *k*- dimensional vector functional $\boldsymbol{J}(\boldsymbol{x}) = (J_1(\boldsymbol{x}), J_2(\boldsymbol{x}), \ldots, J_k(\boldsymbol{x}))$, which elements are functions of vector $\boldsymbol{x}$. Solution of this task is $\boldsymbol{x}^* \in \{X\}$, which is optimal solution of functionals $J_1(\boldsymbol{x}), J_2(\boldsymbol{x}), \ldots, J_k(\boldsymbol{x})$ according to chosen type of vector optimization.

If we connect this definition of vector optimization to complex process P created from partional processes $P_i, i = 1,2, \ldots . k$ (every partial process is defined by equations and constrains and is controlled by its criterion J_i), we can claim that $\boldsymbol{x}^*$ is optimizing process P with regard to function $R(\boldsymbol{x})$ with vector quality pointer $\boldsymbol{J}(\boldsymbol{x})$, if is valid

$$R(\boldsymbol{x}^*) = \inf R(\boldsymbol{x}), with \quad \boldsymbol{x}^* \in \{X\}. \tag{11}$$

From the 70's, when first vector optimization tasks were solved, numbers of methods dealing with this issue were created. We can divide them into two main groups:

- methods defining the set of non improving elements ,
- compromising methods.

A. Methods defining the set of non improving elements

There is no hierarchy of criteria in this group of methods, every criterion is equally important. Task of vector optimization can be defined as trying to minimize the vector $\boldsymbol{J}(\boldsymbol{x}) - \boldsymbol{J}(\boldsymbol{x}_{\boldsymbol{\alpha}})$, where $\boldsymbol{x}_{\boldsymbol{\alpha}}$ is optimal values vector of variables $\boldsymbol{x}$ according to defined criteria.

1) Quadratic norm

The most common criterion used in this norm is minimal sum of quadratic variance of objective functions $J_\alpha(\boldsymbol{x})$ for random $\boldsymbol{x} \in \{X\}$ from objective function $J_\alpha(\boldsymbol{x}_{\boldsymbol{\alpha}})$ for vector of ideal values of chosen criteria $\boldsymbol{x}_{\boldsymbol{\alpha}} = (x_{1\alpha}, x_{2\alpha}, \ldots, x_{n\alpha})$, $\alpha = 1,2, \ldots, k$;

$$R(x) = \sum_{\alpha=1}^{k} (\boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}) - \boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}}))^2$$

$$opt\ R(x) = \min_{x \in \{X\}} \sum_{\alpha=1}^{k} (\boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}) - \boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}}))^2 \tag{12}$$

Values of this norm are usually divided by optimal value. Reason for this is non-dimensional solution value. Calculation formula then looks like:

$$R(x) = \sum_{\alpha=1}^{k} \frac{(\boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}) - \boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}}))^2}{\boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}})^2}$$

$$opt\ R(x) = \min_{x \in \{X\}} \sum_{\alpha=1}^{k} \frac{(\boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}) - \boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}}))^2}{\boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}})^2}. \tag{13}$$

2) Linear norm

It represents minimal sum of linear variance of objective functions $(\boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}))$ from optimal values of objective functions $\boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}})$. Optimal value of functions $R(x)$ can be calculated using formulas:

$$R(x) = \left| \sum_{\alpha=1}^{k} (\boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}) - \boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}}) \right|,$$

$$opt\ R(x) = \min_{x \in \{X\}} \left(\sum_{\alpha=1}^{k} (\boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}) - \boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}}) \right). \tag{14}$$

3) Generalized norm

Objective functions of this norm is

$$R_L(x) = \sum_{\alpha=1}^{k} \left\{ \left((\boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}) - \boldsymbol{J}_{\boldsymbol{\alpha}}(\boldsymbol{x}_{\boldsymbol{\alpha}}) \right)^L \right\}^{\frac{1}{L}}; L \geq 1. \tag{15}$$

For $L = 1$ this functions corresponds to linear norm, for $L = 2$ it corresponds quadratic norm and for $L = \infty$ it is

$$R_\infty(x) = \max_\beta\{(\boldsymbol{J_\alpha}(\boldsymbol{x}) - \boldsymbol{J_\alpha}(\boldsymbol{x_\alpha}); = 1,2,\dots,n\}. \tag{16}$$

All these norms can be refined by multiplying objective function elements with appropriate coefficients. Searched point $\boldsymbol{x}^* \in \{X\}$ is called non improving in space $\{X\}$ regarding to functional $\boldsymbol{J}(\boldsymbol{x})$, if there is no point $\widetilde{\boldsymbol{x}}$ in this space, for which is valid $(\boldsymbol{J_\alpha}(\widetilde{\boldsymbol{x}}) \leq \boldsymbol{J_\alpha}(\boldsymbol{x}^*), \alpha = 1,2,\dots,k$.

B. Compromising methods

These methods are based on defining more strict constrains or adding another constrains to objective functions. Compromise means finding optimal solution by minimizing value of formula $\beta_1 J_1(\boldsymbol{x}) + \beta_2 J_2(\boldsymbol{x}) + \cdots + \beta_k J_k(\boldsymbol{x})$, where β_1 to β_k are carefully chosen importance coefficients. Their values are recommended to be chosen as follows: $\beta_1 = 1/J_{10}$, $\beta_2 = 1/J_{10}$, $\beta_k = 1/J_{k0}$, where J_{k0} are values of objective functions counted in optimization only via selected criterion.

These methods are used when it is possible to define importance of every criterion before starting optimization process, or when additional information about criterion importance is found out during optimization process. Main idea is in defining importance of every scalar criterion. This can affect the result of vector optimization (importance of scalar criterion will be labeled as λ). Most common methods from this group are optimization of weighted sum of scalar criteria and weighted sum of variance vector.

1) Optimization of weighted sum of scalar criteria

Function for this norm is defined by formula

$$opt\ R(x) = opt \sum_{\alpha=1}^{k} \boldsymbol{\lambda_\alpha J_\alpha}(\boldsymbol{x}). \tag{17}$$

Optimal solution is represented by maximum or minimum of weighted sum of scalar criteria, depending on character of problem being solved.

2) Weighted sum of variance vector

In this norm function $R(x)$ is represented by formula

$$R(x) = \sum_{\alpha=1}^{k} \boldsymbol{\lambda_\alpha} * (\boldsymbol{J_\alpha}(\boldsymbol{x}) - \boldsymbol{J_\alpha}(\boldsymbol{x_\alpha}). \tag{18}$$

Optimal solution is always represented by minimum value of function $R(x)$, because this norm is using variation from ideal values.

$$opt\ R(x) = \min_{x \in \{X\}} \left(\sum_{\alpha=1}^{k} \boldsymbol{\lambda_\alpha} * (\boldsymbol{J_\alpha} - \boldsymbol{J_\alpha}(\boldsymbol{x_\alpha}) \right). \tag{19}$$

It is also possible to use quadratic variant of this formula (weighted quadratic norm).[4]

IV. Choosing optimal investment strategy by using vector optimization methods

A. Defining the problem

Investor was given some capital from the lender and he is deciding about the best investment of the funds. He can divide whole capital into three commodities. Investor has economical knowledge of Markowitz model (closely described in [5] and [6]) and his extension for safe investment. At the basis of these knowledge investor defines main conditions of his strategy:

Investor expects minimum profit of 10%. He assumes 10% profitability from investment to 1st commodity, 5% profitability from investment to 2nd commodity a 16% profitability from investment to 3rd commodity. Investor also knows variance σ_i^2 of ith commodity and covariance coefficients σ_{ij} ith and jth commodity. Values of these coefficients are: $\sigma_1^2 = 0{,}2; \sigma_2^2 = 0{,}03; \sigma_3^2 = 0{,}18$; $\sigma_{12} = 0{,}05$; $\sigma_{13} = 0{,}02$; $\sigma_{23} = 0{,}03$. Investor's goal is to minimize the riskiness of his investment. This can be reached by minimizing the variance of commodity portfolio. It is defined by formula $Var_p = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + x_3^2\sigma_3^2 + 2x_1x_2\sigma_{12} + 2x_1x_3\sigma_{13} + 2x_2x_3\sigma_{23}$. [6]

Lender expects refunding of investment by monthly payments. He monthly claims 8% from capital invested to 1st commodity, 10% from capital invested to 2nd commodity and 7% from capital invested to 3rd commodity. Investor also doesn't want to invest more than 60% of given capital to one commodity.

What is the ideal investment strategy for investor in order to minimize the riskiness of investment and also minimize monthly payment to the lender?

B. Solution

This task is vector optimization task. Solution is to find ideal values of vector $\boldsymbol{x}$ by optimizing objective functions

$$0{,}2\,x_1^2 + 0{,}08\,x_2^2 + 0{,}18\,x_3^2 + 0{,}1x_1x_2 + 0{,}04x_1x_3 + 0{,}06x_2x_3 \to min$$
$$0{,}08x_1 + 0{,}1x_2 + 0{,}07x_3 \to min \tag{20}$$

with respecting constrains

$$0{,}1x_1 + 0{,}05x_2 + 0{,}16x_3 \geq 0{,}1\,;$$
$$x_1 + x_2 + x_3 = 1;$$
$$0 \leq x_1, x_2, x_3 \leq 0{,}6. \tag{21}$$

First step is find optimal solutions of objective functions separately, so just for

$$0{,}08x_1 + 0{,}1x_2 + 0{,}07x_3 \to min$$

and

$$0{,}2\,x_1^2 + 0{,}08\,x_2^2 + 0{,}18\,x_3^2 + 0{,}1x_1x_2 + 0{,}04x_1x_3 + 0{,}06x_2x_3 \to min.$$

Result of this optimization is displayed in Table 1. From the results we can claim, that in optimizing for monthly payment investor will invest most of his funds to 3rd commodity and he won't invest any of funds to 2nd one. On the other hand, in minimizing riskiness ratio, most of funds will be invested to 2nd commodity, and the smallest part will be invested to 1st one.

Table 1 Optimizing for separate objective functions

	x_1	x_2	x_3
Optimization for monthly payment	40%	0%	60%
Optimization for riskiness ratio	19,447%	43,938%	36,615%

Another step is calculation of ideal values for both objective functions. This calculation is done by inserting ideal values counted before into objective functions. After this calculation it is clear that minimal monthly payment for lender is 7,4 % from invested funds and minimal portfolio variation is 6,82% (it means that standard deviation is 26,11 %) . After this step the calculation of vector optimization using quadratic norm is performed. Goal of this step is to minimize variation from ideal values of both objective functions:

$$opt\ R(x) = R(x^*) = min \sum_{b=1}^{m} \frac{(J_b(x) - J_b(x_b))^2}{J_b(x_b)^2}. \tag{22}$$

After inserting coefficients form objective functions this equation is created:

$$R(x) = \left(\frac{0{,}08x_1 + 0{,}1x_2 + 0{,}07x_3}{740} - 1\right)^2 +$$
$$+\left(\frac{0{,}2\,x_1^2 + 0{,}08\,x_2^2 + 0{,}18\,x_3^2 + 0{,}1x_1x_2 + 0{,}04x_1x_3 + 0{,}06x_2x_3}{1363700} - 1\right)^2. \tag{23}$$

After the calculation minimal value of variance is found as well as result of vector optimization (ideal vector $\boldsymbol{x}$ using quadratic norm).

Table 2 Optimal percentage distribution of given capital between commodities

	x_1	x_2	x_3
Optimal values	22,892%	36,573%	40,535%

In this distribution monthly payment represents 8,33 % of lent funds and portfolio variance is 7,17% (standard deviation is 26,78%).

Table 3 displays comparison of vector optimization results using various vector optimization methods, specifically linear norm, 4th power norm, and weighted quadratic norm for different importance of objective functions were used (1st value of importance is for optimization of monthly payment, 2nd one is for optimization of riskiness ratio).

Table 3 Ideal distribution values using other norms of vector optimization

	x_1	x_2	x_3
optimization - monthly payment	40%	0%	60%
optimization - riskiness ratio	19,447%	43,938%	36,615%
VO - quadratic norm	22,892%	36,573%	40,535%
VO - linear norm	19,447%	43,938%	36,615%
VO - 4th power norm	22,373%	37,681%	39,945%
VO - weighted quadratic norm (0,4 ; 0,6)	19,447%	43,938%	36,615%
VO - weighted quadratic norm (0,6 ; 0,4)	26,206%	29,489%	44,305%
VO - weighted quadratic norm (0,7 ; 0,3)	37,315%	5,740%	56,945%

Results are also displayed on the graph in Fig. 1.

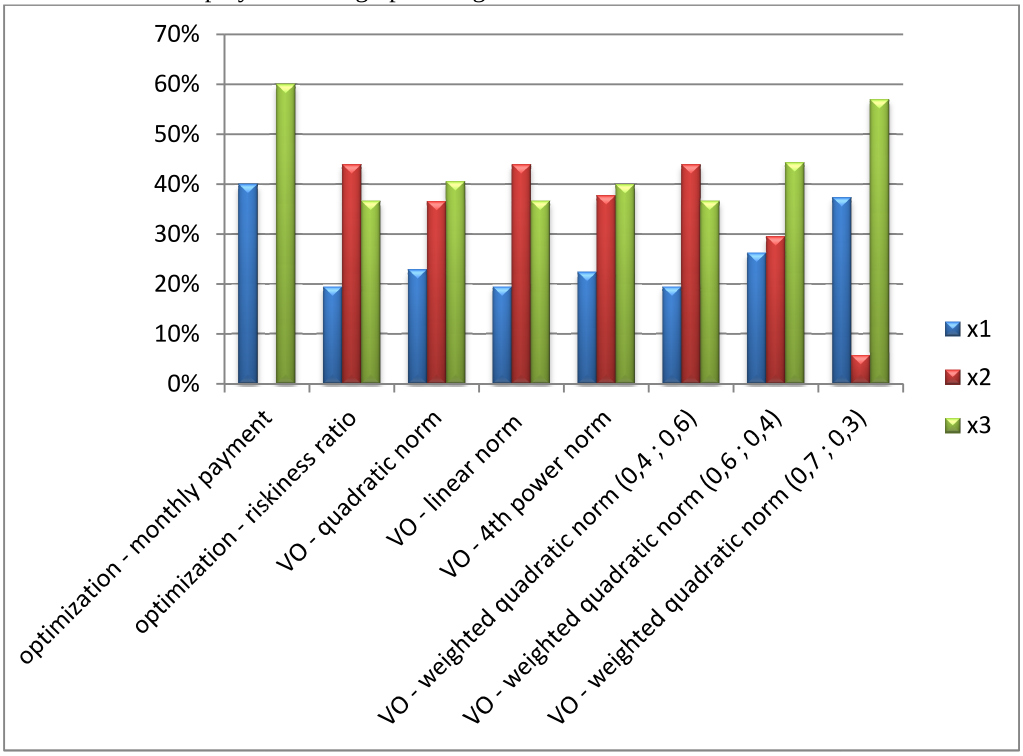

Fig. 1 Percentage of investment distribution into commodities using various methods of vector optimization

Application in MATLAB was created for solving vector optimization tasks. It can be used for solving vector optimization tasks for 2 objective functions with linear or quadratic form. More specific description of this application can be found in [8]. Applications interface is shown in Fig. 2.

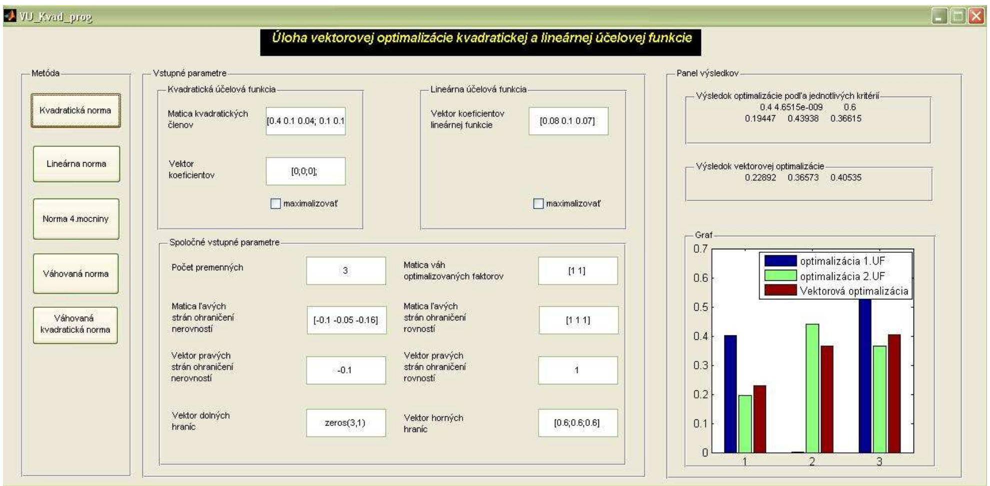

Fig. 2 Application for solving vector optimization problems

V. Conclusion

Vector optimization represents simple and practical way to solve optimization task in case that there is more than one optimized factor. Many organizations face vector optimization problems, mainly when they are trying to reach optimal process functionality in organization. This paper contains various methods of vector optimization. In conclusion we can claim that results of vector optimization mainly depend on chosen method of optimization. In most cases the result of vector optimization represents some kind of compromise between partial optimizations for separate objective functions.

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References

[1] ROSINOVÁ, Danica – DÚBRAVSKÁ Mária: Optimization. Vol.1, Bratislava: STU, 2008. 190 pages. - ISBN: 8022727952.

[2] BEREŽNÝ, Štefan – KRAVECOVÁ, Daniela: Linear programming. Vol.1. Košice: KMTI FEI TU, 2012. 108 s.- ISBN 978-80-553-0910-1.4

[3] JIŘÍČKOVÁ, Viera: Chosen methods of quadratic programming: Diploma thesis. Brno: ÚMaS PF. 2011. 78 pages.

[4] HRUBINA, Kamil - JADLOVSKÁ Anna – HREHOVÁ Stella: Algorithms of optimization methods with using programming systems. Vol.1 . Prešov – Košice, 2005.393 pages. ISBN 80-88941-31-8.

[5] PEJCHAL, Lukáš: Defining optimal weights of funds portfolio: Diploma thesis. Brno: ÚMaS PF. 2011. 73 pages.

[6] INUIGUCHI, Masahiro – TANINO Tetsuzo: Portfolio selection under independent possibilistic information. Fuzzy Sets and Systems, 2000, pages 83-92. Available on intenet:< http://www.geocities.ws/smhurtado/2000-115-1-83-92.pdf>

[7] SKÁCEL, Ján : Riskiness reduction possibilities of portfolio investments: Diploma thesis. Bratislava: KFaE FM UK. 2005. 58 s. Available on internet: < http://diplomovka.sme.sk/zdroj/2737.pdf >.

[8] ČABALA, Ján: Solving economical problems by methods of multiple-criteria decision-making and vector optimization: Diploma thesis. Košice: KKUI FEI TUKE. 2014. 89 pages.