Swing-up and Stabilizing Control of Classical and Rotary Inverted Pendulum Systems

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***Abstract*— The purpose of this paper is to present the design of a complete control strategy for the classical and rotary single inverted pendulum system. The design, which involves swing-up and stabilization of both systems, is based on function blocks and GUI tools from the *Inverted Pendula Modeling and Control*, a *Simulink* block library developed by the author of the paper.**

***Keywords*—classical single inverted pendulum, rotary single inverted pendulum, energy-based swing-up methods, state-feedback stabilization, custom Simulink block library**

I. Introduction

Inverted **p**endula **s**ystems (IPS) represent a significant class of nonlinear underactuated mechanical systems, well-suited for the verification and practice of ideas emerging in control theory and robotics. Stabilization of a pendulum rod in the unstable upright position is considered a benchmark control problem which has been solved by attaching the pendulum to a base that either moves in a controlled linear manner (*classical IPS*) or in a rotary manner in a horizontal plane (*rotary IPS*).

As an example of task-oriented control of IPS, swinging the pendulum up from the pendant to upright position was introduced in [1]. Any system of inverted pendula is therefore a suitable testbed for illustrating hybrid control approaches.

The *Inverted Pendula Modeling and Control (IPMaC)* is a structured *Simulink* block library which was developed by the author of this paper and provides complex software support for the analysis and control of both classical and rotary IPS [2]. Strong emphasis is placed on the generalized approach to system modeling [3], allowing the library to handle systems which differ by the number of pendulum links attached to the base, such as single [2][3], double [2][4] and triple IPS.

This paper aims to present a control strategy which ensures successful swinging up and stabilization of classical and rotary single IPS. Each step of the design relies on suitable function blocks or GUI tools from the *IPMaC*. Since the stabilization problem has already been thoroughly dealt with (see [2][3][4]), the paper will focus on the swing-up design, notably on the comparison of several proposed methods.

II. Mathematical Modeling of Single Inverted Pendulum Systems – Automatic Approach

The *classical single inverted pendulum system* is composed of a rigid, homogenous pendulum rod hinged to a stable mechanism (cart) [5] which allows for free movement along

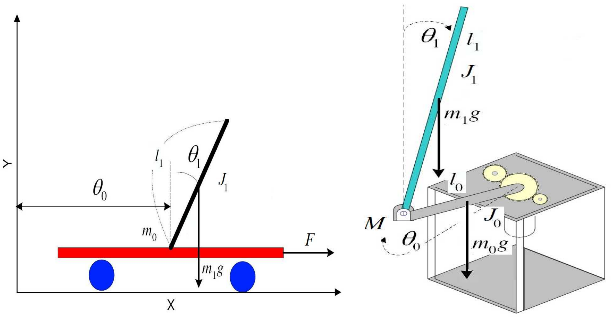

Fig. 1 Scheme and nomenclature: a) Classical single inverted pendulum system b) Rotary single inverted pendulum system

a single axis (Fig. 1a). The *rotary single inverted pendulum system* (i.e. the *Furuta pendulum* [1]) consists of a pendulum rod attached to an arm which is free to rotate in a horizontal plane (Fig. 1b) [6]. Both systems are underactuated: the only input (force applied on the cart or torque applied on the arm) is in each case used to control the two degrees of freedom of the system: base position $\theta_0(t)$ (cart position [m] or arm angle [rad]) and pendulum angle $\theta_1(t)$ [rad].

A. Modeling and Simulation of Considered Systems

The *Inverted Pendula Model Equation Derivator* (Fig. 2) is a MATLAB GUI application and a central component of the *IPMaC*. The application generates the motion equations for the user-chosen types of IPS (classical/rotary, single/double) using an original procedure of mathematical model derivation for the generalized (*n-link*) IPS, which was implemented in MATLAB using *Symbolic Math Toolbox* [3].

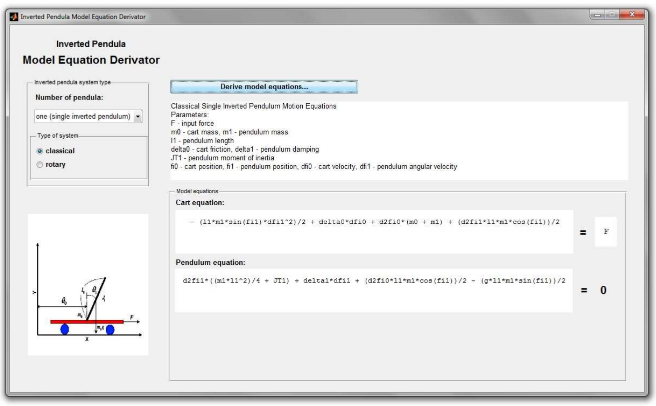

Fig. 2 *Inverted Pendula Model Equation Derivator*

With aid of the *Derivator*, mathematical models of both considered systems were obtained and will be presented below in the rearranged, so-called *standard minimal ODE (ordinary differential equation)* matrix form:

$$\boldsymbol{M}\big(\boldsymbol{\theta}(t)\big)\ddot{\boldsymbol{\theta}}(t)+\boldsymbol{N}\big(\boldsymbol{\theta}(t),\dot{\boldsymbol{\theta}}(t)\big)\dot{\boldsymbol{\theta}}(t)+\boldsymbol{P}\big(\boldsymbol{\theta}(t)\big)=\boldsymbol{V}(t) \qquad (1)$$

where $\boldsymbol{\theta}(t)=\big(\theta_0(t)\quad \theta_1(t)\big)^T$. This approach allows to isolate $\ddot{\boldsymbol{\theta}}(t)$ and express both systems in the nonlinear state-space form

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{f}\big(\boldsymbol{x}(t),u(t),t\big),\\ \boldsymbol{y}(t) &= \boldsymbol{g}\big(\boldsymbol{x}(t),u(t),t\big)\end{aligned} \qquad (2)$$

by defining the state vector as $\boldsymbol{x}(t)=\big(\boldsymbol{\theta}(t)\quad \dot{\boldsymbol{\theta}}(t)\big)^T$.

After the rearrangement, the two second-order nonlinear differential equations of the *classical single inverted pendulum system*, which respectively correspond to the cart and the pendulum, assumed the form:

$$\begin{pmatrix} m_0+m_1 & \frac{1}{2}m_1l_1\cos\theta_1(t) \\ \frac{1}{2}m_1l_1\cos\theta_1(t) & J_1\end{pmatrix}\begin{pmatrix}\ddot\theta_0(t)\\ \ddot\theta_1(t)\end{pmatrix} + \begin{pmatrix}\delta_0 & -\frac{1}{2} m_1l_1\dot\theta_1(t)\sin\theta_1(t)\\ 0 & \delta_1\end{pmatrix}\begin{pmatrix}\dot\theta_0(t)\\ \dot\theta_1(t)\end{pmatrix} + \begin{pmatrix}0\\ -\frac{1}{2} m_1gl_1\sin\theta_1(t)\end{pmatrix} = \begin{pmatrix}F(t)\\0\end{pmatrix} \qquad (3)$$

where m_0 is the cart mass, m_1 is the pendulum mass, l_1 is the pendulum length, δ_0 is the friction coefficient of the cart against the rail, δ_1 is the damping constant in the joint of the pendulum, $J_1=\frac{1}{3}m_1l_1^2$ is the pendulum's moment of inertia with respect to the pivot point and $F(t)$ is the force induced on the cart.

Analogically, the mathematical model of the *rotary inverted pendulum system*, composed of two second-order nonlinear differential equations which respectively describe the rotary arm and the pendulum, became:

$$\begin{pmatrix} J_0+m_1l_0^2+\frac{1}{4}m_1l_1^2\sin^2\theta_1(t) & \frac{1}{2}m_1l_0l_1\cos\theta_1(t) \\ \frac{1}{2}m_1l_0l_1\cos\theta_1(t) & J_1\end{pmatrix}\begin{pmatrix}\ddot\theta_0(t)\\ \ddot\theta_1(t)\end{pmatrix} + \begin{pmatrix}\delta_0+\frac{1}{4}m_1l_1^2\dot\theta_1(t)\sin 2\theta_1(t) & -\frac{1}{2}m_1l_0l_1\dot\theta_1(t)\sin\theta_1(t)\\ -\frac{1}{8}m_1l_1^2\dot\theta_0(t)\sin 2\theta_1(t) & \delta_1\end{pmatrix}\begin{pmatrix}\dot\theta_0(t)\\ \dot\theta_1(t)\end{pmatrix} + \begin{pmatrix}0\\ -\frac{1}{2}m_1gl_1\sin\theta_1(t)\end{pmatrix} = \begin{pmatrix}M(t)\\0\end{pmatrix} \qquad (4)$$

where m_0, m_1 stand for the masses of the arm and the pendulum, l_0, l_1 are their respective lengths, δ_0, δ_1 are the damping constants in the joints of the arm and pendulum, $J_0=\frac{1}{3}m_0l_0^2$ and $J_1=\frac{1}{3}m_1l_1^2$ are the moments of inertia of the arm and pendulum with respect to their pivot points and $M(t)$ is the input torque applied upon the rotary arm.

Both models listed above (hereafter referred to as *force/torque models*) are among those which were included in the *Inverted Pendula Models* sublibrary of the *IPMaC* [2] in form of atomic library blocks: the *Classical Single Inverted Pendulum* and the *Rotary Single Inverted Pendulum* block.

B. *Modeling and Simulation of Actuating Mechanisms*

To provide the simulation models of IPS with a model of the most frequently used actuating mechanism, a library block *DC Motor for Inverted Pendula Systems* was included into the the *Inverted Pendula Motors* sublibrary of the *IPMaC*. The block implements the mathematical model of a brushed direct-current (DC) motor in two alternative forms depending on the type of system (classical / rotary) [7]:

- a voltage-to-force conversion relationship:

$$F(t)=\frac{k_mk_g}{R_ar}V_a(t)-\frac{k_m^2k_g^2}{R_ar^2}\dot\theta_0(t) \qquad (5)$$

- and a voltage-to-torque conversion relationship:

$$M(t)=\frac{k_mk_g}{R_a}V_a(t)-\frac{k_m^2k_g^2}{R_a}\dot\theta_0(t) \qquad (6)$$

where $V_a(t)$ is the input voltage applied to the motor, k_m is the motor torque constant, equal in value to the back EMF constant, k_g is the gear ratio, R_a is the armature resistance and r is the radius of the tooth pulley which is coupled to the motor shaft and converts the torque produced by the motor into the linear driving force $F(t)$ (hence, r is only necessary for classical IPS). If the DC motor model is appended to an inverted pendulum system (i.e. (5) is substituted into (3), or (6) is substituted into (4)), a *voltage model* of the system is obtained.

III. Outline of a Control Strategy for Swing-up and Stabilization of Single Inverted Pendulum Systems

As a principal control objective, the pendulum had to be *swung up from the stable downward to the unstable upright equilibrium* ($\boldsymbol{x}(t)=\boldsymbol{0}^T$), *captured and stabilized there* [8]. This problem leads to a hybrid control setup which consists of three basic components, schematically depicted in the block diagram in Fig. 3:

- *a swing-up controller*, which actuates the base with an input signal of appropriate direction and magnitude, making the pendulum swing with an increasing amplitude and angular speed until it enters the stabilization zone (balancing region) around the upright position [1],
- *a stabilizing (balancing) controller*, which maintains the pendulum in the upright position using suitable linear or nonlinear feedback control techniques [2][4][5][9],
- *a transition (switching) mechanism*, which intercepts the pendulum when it nears the upright position (crosses the borderline of the balancing region) and switches to stabilizing control [8].

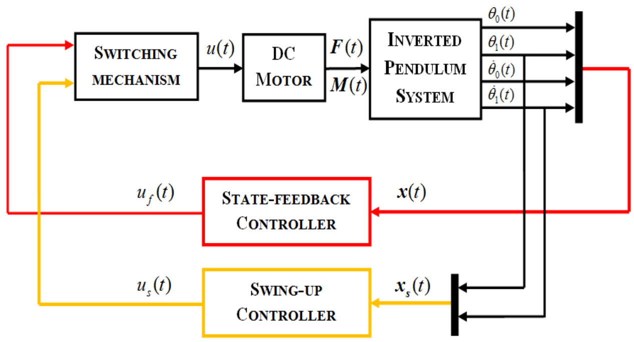

Fig. 3 Block diagram of swing-up and stabilization control of IPS

A. *Energy-Based Methods for Swing-up Control*

One of the earliest and most effective approaches to pendulum swing-up is the method based on energy considerations, proposed and explained by Åström, Furuta and Iwase in [8][10]. The goal is to maximize the total mechanical energy W of the pendulum in the upright position. The first derivative of the energy is given as

$$\dot{W} = \frac{d}{dt}\left(\frac{1}{2}ml^2\dot{\theta}_1^2(t) + mgl\left(1-\cos\theta_1(t)\right)\right) = ml\dot{\theta}_1(t)\cos\theta_1(t) \,. \quad (7)$$

Whether is greater or lower than zero depends directly on the sign of $\dot{\theta}_1(t)\left(\cos\theta_1(t)\right)$. The basic energy-based control law (*cosine value controller*) therefore becomes [11]:

$$u(t) = -u_m \operatorname{sgn}\left(\dot{\theta}_1(t)\cos(\theta_1(t))\right) \quad (8)$$

This control law can be simplified into a law which will be referred to as a *zero speed controller*:

$$u(t) = u_m \operatorname{sgn}\left(\dot{\theta}_1(t)\right) \quad (9)$$

or modified into an *absolute value* control law:

$$u(t) = u_m \operatorname{sgn}\left(\dot{\theta}_1(t)\left|\theta_1(t)\right|\right) \quad (10)$$

The control laws (8)-(10) were encapsulated into the structure of *Swing-up Controller* block of the *IPMaC*, which allows the user to select a swing-up method, the constraints of the balancing region and the input magnitude u_m.

B. *Stabilization via State-Feedback Control Techniques*

Once the pendulum has been captured in the upright position, we need to switch to a balancing controller. The *Inverted Pendula Control* sublibrary of the *IPMaC* provides complex software support for the *linear state-feedback (S-F) controller design*. The blocks it contains (*S-F Controller with Feedforward Gain*, *S-F Controller with Integral Action* and *Luenberger Estimator*, among others) were thoroughly described in [2][4] in terms of their structure and functionality.

With aid of the mentioned blocks, the feedback gain vector which brings a linear system into the state space origin can be determined alternatively through pole-placement and linear quadratic regulator (LQR), using either the continuous-time or discrete-time linearized state-space models of IPS. These can be obtained from the *Inverted Pendula Model Linearizator & Discretizer* (Fig. 4), a MATLAB GUI application which generates the state-space matrices by expanding (2) into the Taylor series. Additional control structures ensure that the output reaches a reference position by means of feedforward gain and any permanent steady-state error is eliminated by the integral control.

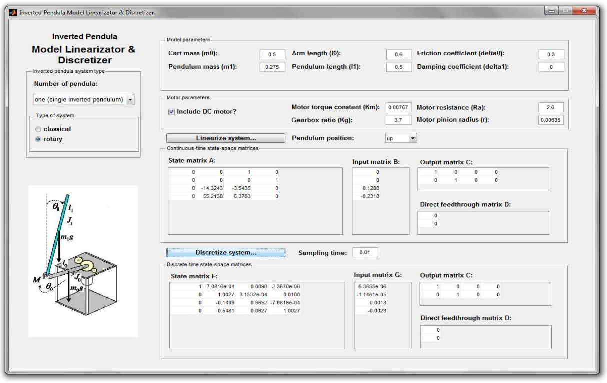

Fig. 4 *Inverted Pendula Model Linearizator & Discretizer*

IV. Evaluation of the Control Strategy

The proposed control strategy was verified for the *force/torque models* and *voltage models* of both classical single and rotary single IPS. The numeric parameters were chosen to be $m_0 = 0.5kg$, $m_1 = 0.275kg$, $l_0 = 0.6m$, $l_1 = 0.5m$, $\delta_0 = 0.3kgs^{-1}$, $\delta_1 = 0kgm^2s^{-1}$, and the initial conditions were set to $\theta_0(0) = -1$ and $\theta_1(0) = -\pi$. The DC motor model parameters were borrowed from the motor featured within the series of popular laboratory models of IPS issued by *Quanser Academic* [7]. The weight matrices of the standard discrete-time LQ functional $J_{LQR}(i) = \sum_{i=0}^{N-1} \mathbf{x}^T(i)\mathbf{Q}\mathbf{x}(i) + u^T(i)ru(i)$ for the balancing state-feedback controller were specified as $Q = diag(100 \quad 20 \quad 20 \quad 0)$, $r = 1$ for every simulation.

Fig. 5 and Fig. 6 depict the comparison of the three swing-up control laws (8)-(10) applied on the force/torque models of IPS. For each system and swing-up control law, the input magnitude was tuned to the highest value which still allows successful swing-up with no oscillations or destabilization (Tab. 1). Switching between the controllers took place when the pendulum was 0.6 rad away from the upright position.

TABLE I
Highest input magnitudes which ensure successful swing-up

Type of swing-up controller	classical single inverted pendulum (force [N])	rotary single inverted pendulum (torque [Nm])
absolute value controller	2,1	2,2
zero speed controller	2,6	2
cosine value controller	1,6	1,3

In the case of *classical inverted pendulum*, the best performance in terms of both pendulum swing-up time and

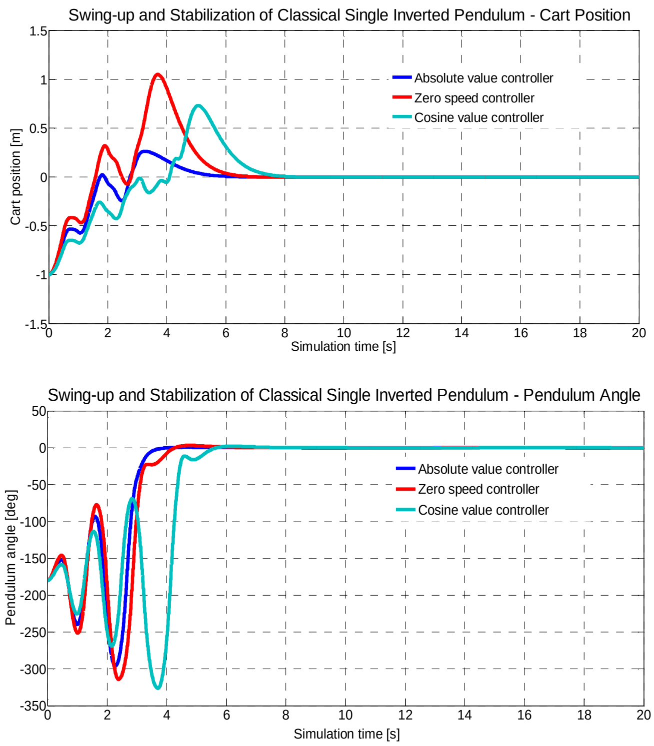

Fig. 5 Swing-up and stabilization of the classical single inverted pendulum force model – comparison of methods

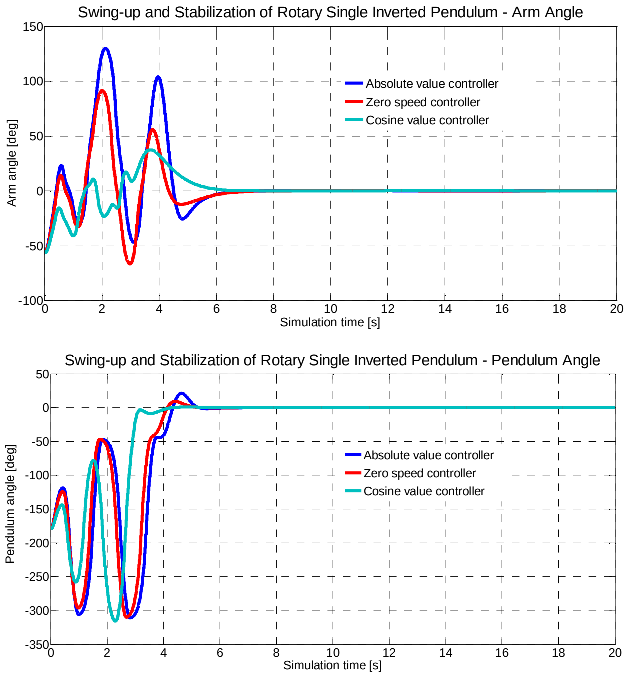

Fig. 6 Swing-up and stabilization of the rotary single inverted pendulum torque model – comparison of methods

total displacement of the cart was provided by the absolute-value controller. The zero speed controller only managed to stabilize the pendulum at the comparable time at the price of a large cart overshoot. The *rotary inverted pendulum* performed at its best for the cosine-value controller – the performance was diminished for other controllers due to long intervals during which energy is taken from the pendulum.

When tuning the input magnitude u_m, the time needed to swing the pendulum to the upright position was discovered as indirectly proportional to u_m. However, feasible input values could only be selected from a bounded interval. An unlimited increase in magnitude leads to the increase of the pendulum

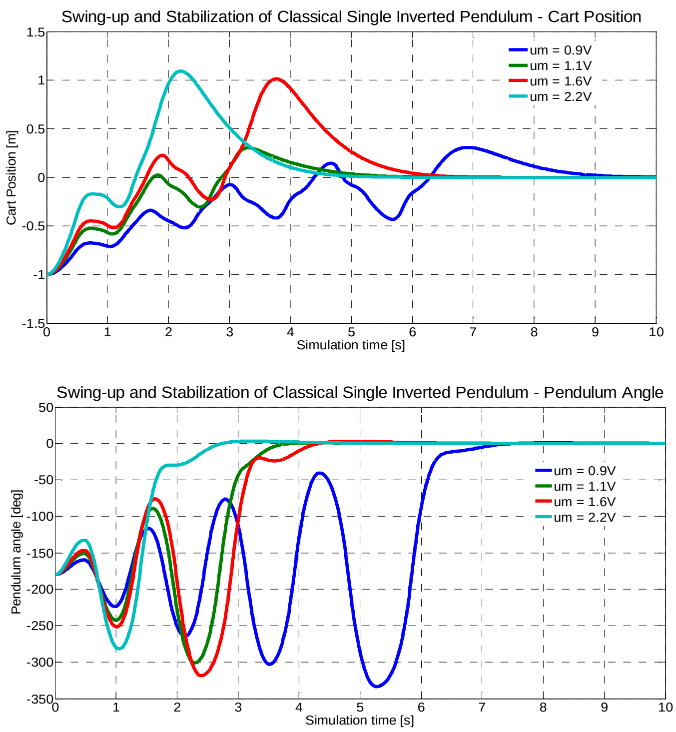

Fig. 7 Swing-up and stabilization of the classical single inverted pendulum voltage model – comparison of the effect of varying input magnitude

angular speed, and a too low magnitude may either be entirely unable to bring the pendulum upright, or it needs numerous swings to do so, which results in a displacement of the base away from the equilibrium point. In both cases, the control mechanism would fail to stabilize the system, proving that the balancing controller is only effective if the state of system is close to the equilibrium point in the moment of interception. An illustrative experiment which compares the effect of a varying u_m on the classical single inverted pendulum voltage model, is depicted in Fig. 7.

V. Conclusion

The purpose of this paper was to provide a thorough overview of the problem of swing-up and stabilizing control design for one-link inverted pendula systems. Several swing-up methods based on energy considerations were proposed and compared in simulation experiments involving models of a classical and rotary single inverted pendulum; linear state-feedback control techniques were next used to stabilize the pendulum, captured in the upright position. Most importantly, it was shown that the *Inverted Pendula Modeling and Control*, the custom-designed *Simulink* block library which was used as a software framework for all issues covered in the paper, is a highly suitable program tool for design and verification of hybrid control methods for nonlinear systems.

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