Application of the State-Dependent Riccati Equation Method in Nonlinear Control Design for Inverted Pendulum Systems

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Abstract— **The purpose of this paper is to present the application of the nonlinear control design technique based on state-dependent Riccati equation (SDRE) on the generalized (n-link) inverted pendulum system. All relevant steps of the control design process, which include presenting the system structure, transformation of the system description into a specific equivalent form, and deriving the discrete-time SDRE formulation, are described in detail. The results of the proposed control design technique are demonstrated on a simulation model of the rotary single inverted pendulum system using appropriate blocks and applications from the Simulink block library developed by the authors of the paper. The advantages of a SDRE-based control algorithm over standard linear quadratic optimal control are shown by comparing both methods in simulation experiments which involve stabilization of the pendulum in the unstable upright equilibrium.**

I. Introduction

In recent years, numerous methodologies of stabilization and tracking control design for nonlinear underactuated systems have been studied. These include linear design techniques based on Jacobian linearization, gain scheduling, sliding mode control and partial feedback linearization, as well as linear/nonlinear adaptive control [1][2]. One of the systematic and effective ways for nonlinear state-feedback control design for underactuated systems is the method based on the solution of a *state-dependent Riccati equation* (*SDRE*), which relies on the original nonlinear state-space description of the system and creates a separate linear quadratic optimal control problem (LQR) at each time step. As a result, tradeoff between acceptable control accuracy (state error) and control input effort is ensured, which is a property not generally found in other nonlinear control design methods [3].

As a significant class of nonlinear unstable underactuated mechanical systems, **i**nverted **p**endulum **s**ystems (IPSs) are well-suited for verification and practice of ideas and techniques emerging in control theory and robotics [4]. Stabilization of a set of interconnected pendulum links in the unstable upright position is considered a benchmark control problem which has been solved by attaching the pendulum links to a base that moves in a controlled linear manner (*classical or linear IPSs*) or in a rotary manner in a horizontal plane (*rotary IPSs*) [5]. Principles of modeling and control of IPSs can further be considered as the basic starting point for the research of advanced underactuated systems such as mobile robots and manipulators, as well as aircraft and watercraft vehicles [6].

The aim of this paper is to provide a methodological summary of the SDRE control design technique with references to the *generalized inverted pendulum system*. The introduction of this concept in [5] allows to treat an arbitrary system of interconnected inverted pendula as a particular instance of the system of n pendula attached to a given stabilizing base. The paper is organized as follows. After presenting the system in the general state-space form in Section 2, discrete-time SDRE formulation is derived based on the discrete-time LQR algorithm in Section 3. Stabilizing results of the SDRE-based control design method are demonstrated in Section 4 using the rotary single inverted pendulum simulation model as a testbed system, and they are concurrently compared with the results of a discrete-time LQR control algorithm. The approached control objectives include balancing the pendulum around the upright position following a variety of initial pendulum deflections, and tracking a predefined reference trajectory by the rotary arm while keeping the pendulum upright. All simulation experiments are conducted using suitable blocks or GUI tools from the *Simulink* block library developed by the authors of the paper – *Inverted Pendula Modeling and Control* (*IPMaC*), which provides complex software support for the analysis and control of both classical and rotary IPSs [7].

II. Generalized (N-Link) Inverted Pendulum System – An Overview

The generalized *system of classical inverted pendula* was introduced in [5] as a set of $n \geq 1$ rigid, homogenous, isotropic rods (*pendulum links*) which are interconnected in joints and attached to a cart (i.e. a stable mechanism which allows for movement alongside a single axis). Analogically, the generalized *system of rotary inverted pendula* was assumed to be composed of $n \geq 1$ interconnected homogenous rods mounted on a rigid arm which rotates in a horizontal plane, perpendicular to the pendula [5]. In both cases, every possible configuration of the system can be uniquely defined by the following vector of generalized coordinates which correspond to the $n+1$ degrees of freedom (DoFs) of a multi-body system:

$$\theta(t) = \begin{pmatrix} \theta_0(t) & \theta_1(t) & \dots & \theta_n(t) \end{pmatrix}^T \qquad (1)$$

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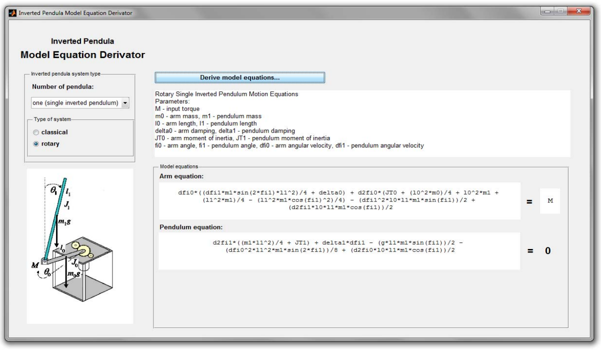

Figure 1. Inverted Pendula Model Equation Derivator

where $\theta_0(t)$ represents the position of the base (cart position or arm angle) and $\theta_1(t)$ to $\theta_n(t)$ stand for the angles of pendulum links.

The dynamic behavior of an arbitrary IPS is consequently described by a mathematical model in the form of $n+1$ nonlinear second-order ordinary differential equations of motion that respectively describe the base and each pendulum link. With the development of the *Inverted Pendula Model Equation Derivator* (Fig. 1), a MATLAB GUI application and a central component of the *IPMaC* [7], it has become possible to effortlessly obtain the motion equations for any system of a desired type (*classical/rotary*) with a defined number of pendulum links. The application implements a general algorithmic procedure of mathematical model derivation which yields the *Euler-Lagrange equations* of motion, defined as

$$\frac{d}{dt}\left(\frac{\partial L(t)}{\partial \dot{\boldsymbol{\theta}}(t)}\right)-\frac{\partial L(t)}{\partial \boldsymbol{\theta}(t)}+\frac{\partial D(t)}{\partial \dot{\boldsymbol{\theta}}(t)}=\boldsymbol{Q}^{*}(t) \tag{2}$$

for a user-specified instance of a generalized classical and rotary IPS. The underlying physics of the procedure was presented in [5] together with generated motion equations of example IPSs and verification of their validity.

Every system of classical or rotary inverted pendula is a *nonlinear, underactuated, time-invariant, controllable SIMO system* with a degree of $2n+2$. The state-space description of such a dynamic system is composed of a differential vector state equation $\boldsymbol{f}(\bullet)$ and an algebraic output equation $\boldsymbol{h}(\bullet)$ [1][8]:

$$\begin{aligned}\dot{\boldsymbol{x}}(t)&=\boldsymbol{f}\left(\boldsymbol{x}(t),u(t)\right)\\ \boldsymbol{y}(t)&=\boldsymbol{h}\left(\boldsymbol{x}(t),u(t)\right)\end{aligned} \tag{3}$$

given that the state vector $\boldsymbol{x}(t)\in\mathbb{R}^{(2n+2)\times 1}$ is by default introduced in the following form:

$$\boldsymbol{x}(t)=\left(\boldsymbol{\theta}(t)\quad \dot{\boldsymbol{\theta}}(t)\right)^{T}=\left(x_1(t)\quad x_2(t)\quad \dots\quad x_{2n+2}(t)\right)^{T} \tag{4}$$

the only input of the system $u(t)\in\mathbb{R}$ is identified either with the applied force or the torque, depending on the type of the base:

$$u(t)=F(t)\quad u(t)=M(t) \tag{5}$$

and the output vector $\boldsymbol{y}(t)\in\mathbb{R}^{m\times 1}$ denotes the m state variables which are assumed to be available through measurement.

For the purposes of further model analysis as well as control design, it is convenient to rearrange the generated nonlinear equations into the so-called standard *(minimal ODE – ordinary differential equation*) form given as:

$$\boldsymbol{M}\left(\boldsymbol{\theta}(t)\right)\ddot{\boldsymbol{\theta}}(t)+\boldsymbol{N}\left(\boldsymbol{\theta}(t),\dot{\boldsymbol{\theta}}(t)\right)\dot{\boldsymbol{\theta}}(t)+\boldsymbol{R}\left(\boldsymbol{\theta}(t)\right)=\boldsymbol{V}(t)u(t) \tag{6}$$

where $\boldsymbol{M}\left(\boldsymbol{\theta}(t)\right)$ is the inertia matrix, $\boldsymbol{N}\left(\boldsymbol{\theta}(t),\dot{\boldsymbol{\theta}}(t)\right)$ describes the influence of centrifugal and Coriolis forces, $\boldsymbol{R}\left(\boldsymbol{\theta}(t)\right)$ accounts for gravity forces and $\boldsymbol{V}(t)$ is the system's input vector [9]. This way, it is possible to isolate the second derivative of the vector of generalized coordinates and subsequently obtain:

$$\begin{pmatrix}\dot{\boldsymbol{\theta}}(t)\\ \ddot{\boldsymbol{\theta}}(t)\end{pmatrix}=\begin{pmatrix}\dot{\boldsymbol{\theta}}(t)\\ \left(\boldsymbol{M}(\boldsymbol{\theta}(t))\right)^{-1}\left(\boldsymbol{V}(t)u(t)-\boldsymbol{N}\left(\boldsymbol{\theta}(t),\dot{\boldsymbol{\theta}}(t)\right)\dot{\boldsymbol{\theta}}(t)-\boldsymbol{R}\left(\boldsymbol{\theta}(t)\right)\right)\end{pmatrix} \tag{7}$$

All elements of (7) can obviously be substituted by their $\boldsymbol{x}(t)$ counterparts, which yields a state equation of the system in the following equivalent form [10]:

$$\dot{\boldsymbol{x}}(t)=\begin{pmatrix}\boldsymbol{0} & \boldsymbol{I}\\ \boldsymbol{0} & \boldsymbol{\text{-}M}^{-1}\boldsymbol{N}\end{pmatrix}\boldsymbol{x}(t)+\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{\text{-}M}^{-1}\boldsymbol{R}\end{pmatrix}+\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{\text{-}M}^{-1}\boldsymbol{V}\end{pmatrix}u(t)\,. \tag{8}$$

If the upright position of all pendulum links corresponds to the equilibrium point of $\boldsymbol{x}(t)=\boldsymbol{x}_S=\boldsymbol{0}^T$ and the system input is set to $u(t)=u_S=0$ [7][8], then the continuous-time state-space description of the linear, time-invariant system which serves as a Taylor approximation of a nonlinear model of an IPS in the upright equilibrium becomes:

$$\begin{aligned}\dot{\boldsymbol{x}}(t)&=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{b}u(t)\\ \boldsymbol{y}(t)&=\boldsymbol{C}x(t)+du(t)\end{aligned} \tag{9}$$

where $\boldsymbol{A}\in\mathbb{R}^{(2n+2)\times(2n+2)}$ is the state (*Jacobian*) matrix of the linearized system, $\boldsymbol{b}\in\mathbb{R}^{(2n+2)\times 1}$ is the input matrix, $\boldsymbol{C}\in\mathbb{R}^{m\times(2n+2)}$ is the output matrix and $d\in\mathbb{R}^{m\times 1}$ is the direct feedthrough matrix [11]. Given the continuous-time state-space description of a *SIMO* system in the form (9), the corresponding discrete-time state-space description is:

$$\begin{aligned}\boldsymbol{x}(i+1)&=\boldsymbol{F}\boldsymbol{x}(i)+\boldsymbol{g}u(i)\\ \boldsymbol{y}(i)&=\boldsymbol{C}x(i)+du(i)\end{aligned} \tag{10}$$

where $\boldsymbol{F}\in\mathbb{R}^{(2n+2)\times(2n+2)}$ is the discrete state matrix, $\mathbf{g}\in\mathbb{R}^{(2n+2)\times 1}$ is the discrete input matrix; $\boldsymbol{C}$ and d are the same as in the continuous system.

III. OPTIMAL STATE-FEEDBACK CONTROL DESIGN FOR INVERTED PENDULUM SYSTEMS

For an arbitrary nonlinear underactuated IPS, the principal control objective is defined as *stabilization of all pendulum links in the vertical upright (inverted) unstable position* following an initial deflection of each pendulum link (i.e. nonzero initial conditions), or a time-constrained (impulse) disturbance input signal. Moreover, stabilization of the pendulum links has to be constantly ensured while the base position is tracking a desired reference command signal [7][8][12].

In this section, two optimal control techniques will be approached: standard discrete-time linear quadratic optimal (LQR) control design which requires the discrete-time linearized state-space model of the system [8][11], and a nonlinear control strategy based on the transformation of the original nonlinear system into a specific equivalent form (obtained through *extended linearization*), which allows the use of LQR control techniques in every time step [3][13][14].

A. Linear Quadratic Regulator (LQR) Design Based on a Discrete-Time State-Space Model

The goal of optimal control design for a linear, time-invariant dynamic system is to determine such feedback control so that a given criterion of optimality is achieved [11][15]. In case the considered linear system is actually a linear approximation of a nonlinear system around a given equilibrium point (as is the case specified in Section 2), then the optimal control techniques designed for linear systems yield an approximate, locally near-optimal stabilizing solution to the problem with guaranteed closed-loop stability and robustness.

Assuming a discrete-time linear system (10) which represents the linear approximation of an arbitrary IPS, the goal of optimal LQR control design is to determine such a state-feedback control vector $\boldsymbol{k}_{LQR}(i)$ so that the resulting feedback control law, defined as

$$u(i) = -\boldsymbol{k}_{LQR}(i)\,\boldsymbol{x}(i) \tag{11}$$

would minimize the total value of an accumulative cost function $J_{LQR}(i)$ over a horizon of *N* time steps [15]:

$$J_{LQR}(i) = \sum_{k=0}^{N-1} \boldsymbol{x}^T(i)\boldsymbol{Q}\boldsymbol{x}(i) + u^T(i)\,r\,u(i) \tag{12}$$

assuming that $\boldsymbol{Q} \in \mathbb{R}^{(2n+2)\times(2n+2)}$ is a positive semidefinite matrix, $r \in \mathbb{R}$ is a positive definite matrix and (12) is subject to first-order time-invariant dynamic constraints, represented by the discrete-time linear dynamic model (10). The difference dynamic Riccati equation is solved iteratively backwards to obtain the optimal sequence of solutions $\boldsymbol{P}(i)$:

$$\boldsymbol{P}(i-1) = \boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{F} - \boldsymbol{F}^T\boldsymbol{P}(i)\boldsymbol{g}\left(\boldsymbol{r} + \boldsymbol{g}^T\boldsymbol{P}(i)\boldsymbol{g}\right)^{-1}\boldsymbol{g}^T\boldsymbol{P}(i)\boldsymbol{F} + \boldsymbol{Q} \tag{13}$$

and the optimal sequence of feedback gain vectors is hence given as

$$\boldsymbol{k}_{LQR}(i) = \left(\boldsymbol{r} + \boldsymbol{g}^T\boldsymbol{P}(i)\boldsymbol{g}\right)^{-1}\boldsymbol{g}^T\boldsymbol{P}(i)\boldsymbol{F}\,. \tag{14}$$

The positive definite steady-state solution $\boldsymbol{P}$ to the difference Riccati equation is obtained by solving the discrete-time algebraic Riccati equation

$$\boldsymbol{F}^T\boldsymbol{P}\boldsymbol{F} - \boldsymbol{F}^T\boldsymbol{P}\boldsymbol{g}\left(\boldsymbol{r} + \boldsymbol{g}^T\boldsymbol{P}\boldsymbol{g}\right)^{-1}\boldsymbol{g}^T\boldsymbol{P}\boldsymbol{F} + \boldsymbol{Q} = \boldsymbol{0} \tag{15}$$

and the optimal solution to the discrete-time LQR problem is computed as

$$\boldsymbol{k}_{LQR} = \frac{1}{r}\boldsymbol{g}^T\boldsymbol{P} \tag{16}$$

where $\boldsymbol{k}_{LQR}$ is the constant optimal feedback gain vector which brings the state vector $\boldsymbol{x}(i)$ into the origin of the state space and the corresponding discrete-time control law assumes the form

$$u(i) = -\boldsymbol{k}_{LQR}\boldsymbol{x}(i)\,. \tag{17}$$

In order to ensure that the output of the IPS will track the predefined reference command $w(i)$, the discrete-time control law was expanded by a feedforward component:

$$u(i) = -\boldsymbol{k}_{LQR}\boldsymbol{x}(i) + k_{ffLQR}w(i) \tag{18}$$

where k_{ffLQR} is the feedforward gain computed in the form

$$k_{ffLQR} = \frac{1}{\boldsymbol{c}_1\left(\boldsymbol{I} - \left(\boldsymbol{F} - \boldsymbol{g}\boldsymbol{k}_{LQR}\right)\right)^{-1}\boldsymbol{g}}\,, \tag{19}$$

and vector $\boldsymbol{c}_1 = (1\ \ 0\ \ \dots\ \ 0)$ is introduced to isolate the base position for tracking control purposes [8][15].

B. Nonlinear Control Design Based on State-Dependent Riccati Equation Approach (SDRE)

As a nonlinear extension to standard LQR method, the control design approach based on *state-dependent Riccati equation* has garnered considerable attention [13][14][16] as a method which simultaneously preserves the original nonlinear dynamics (3) of the system and allows the use of optimal control techniques designed for linear systems as described in Section 3.1, yielding a locally asymptotically optimal solution in the multivariable case [16].

No linear approximation in an equilibrium point is required if the SDRE method is applied on an arbitrary IPS. Instead, the nonlinear equations of motion are manipulated into an equivalent affine representation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}\left(\boldsymbol{x}(t)\right) + \boldsymbol{g}\left(\boldsymbol{x}(t)\right)u(t) \tag{20}$$

which is then rearranged into a *pseudo-linear state-dependent coefficient (SDC) form*

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\left(\boldsymbol{x}(t)\right)\boldsymbol{x}(t)+\boldsymbol{b}\left(\boldsymbol{x}(t)\right)u(t) \tag{21}$$

in which the system matrices are expressed explicitly as functions of the current state [16]. The conversion of the mathematical model of IPSs from the standard minimal form (6) to the form (21) can only be obtained if there exists a transformation matrix $\boldsymbol{R}_m\left(\boldsymbol{\theta}(t)\right)$ such that the absolute term $\boldsymbol{R}\left(\boldsymbol{\theta}(t)\right)$ of (6) becomes a direct function of the system state:

$$\boldsymbol{R}\left(\boldsymbol{\theta}(t)\right)=\boldsymbol{R_m}\left(\boldsymbol{\theta}(t)\right)\boldsymbol{\theta}(t)\,. \tag{22}$$

Analyses of the standard minimal forms for both classical and rotary IPSs revealed that for a system of n inverted pendula, the $\boldsymbol{R}\left(\boldsymbol{\theta}(t)\right)$ matrix assumes the form

$$\boldsymbol{R}\left(\boldsymbol{\theta}(t)\right)=\begin{pmatrix}0 & -r_1\sin\theta_1(t) & -r_2\sin\theta_2(t) & \ldots & -r_n\sin\theta_n(t)\end{pmatrix}^T \tag{23}$$

where $(r_1, r_2, \ldots, r_n)$ are constant parameter-dependent values. To guarantee the equality (22), the transformation matrix can be defined as the following square matrix:

$$\boldsymbol{R_m}\left(\boldsymbol{\theta}(t)\right)=\begin{pmatrix}0 & 0 & 0 & \cdots & 0\\ 0 & -r_1\dfrac{\sin\theta_1(t)}{\theta_1(t)} & 0 & \cdots & 0\\ 0 & 0 & -r_2\dfrac{\sin\theta_2(t)}{\theta_2(t)} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & -r_n\dfrac{\sin\theta_n(t)}{\theta_n(t)}\end{pmatrix}. \tag{24}$$

As a result, the nonlinear state equation of an arbitrary IPS represented as (6) takes the form:

$$\dot{\boldsymbol{x}}(t)=\begin{pmatrix}\boldsymbol{0} & \boldsymbol{I}\\ \boldsymbol{-M}^{-1}\boldsymbol{R_m} & \boldsymbol{-M}^{-1}\boldsymbol{N}\end{pmatrix}\boldsymbol{x}(t)+\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{-M}^{-1}\boldsymbol{V}\end{pmatrix}u(t) \tag{25}$$

which is equivalent to (8). For every i-th time step, the system is next discretized and the discrete-time state equation becomes

$$\boldsymbol{x}(i+1)=\boldsymbol{F}_i\boldsymbol{x}(i)+\boldsymbol{g}_i u(i) \tag{26}$$

where $\boldsymbol{F}_i=\boldsymbol{F}\left(\boldsymbol{x}(i)\right)$ and $\boldsymbol{g}_i=\boldsymbol{g}\left(\boldsymbol{x}(i)\right)$ are the discrete-time pseudo-linear state-space matrices [9][10]. Since $\boldsymbol{F}_i$ and $\boldsymbol{g}_i$ are *treated as being constant* (although different) at every i-th time step, the feedback control law, optimized for the momentary measured/estimated state of a nonlinear IPS is specified to be:

$$u(i)=-\boldsymbol{k}_{SDRE}\left(\boldsymbol{x}(i)\right)\boldsymbol{x}(i)=-\frac{1}{r}\boldsymbol{g}_i\boldsymbol{P}_i\boldsymbol{x}(i) \tag{27}$$

where $\boldsymbol{P}_i$ is the positive definite steady-state solution of the nonlinear state-dependent difference Riccati equation, obtained for the i-th time step by solving the algebraic discrete-time Riccati equation with pseudo-linear matrices:

$$\boldsymbol{F}_i^T\boldsymbol{P}_i\boldsymbol{F}_i-\boldsymbol{F}_i^T\boldsymbol{P}_i\boldsymbol{g}_i\left(\boldsymbol{r}+\boldsymbol{g}_i^T\boldsymbol{P}_i\boldsymbol{g}_i\right)^{-1}\boldsymbol{g}_i^T\boldsymbol{P}_i\boldsymbol{F}_i+\boldsymbol{Q}=\boldsymbol{0}\,. \tag{28}$$

The extended control law which ensures that the base will maintain the predefined reference position is defined in an analogical form to (18):

$$u(i)=-\boldsymbol{k_{SDRE}}\left(\boldsymbol{x}(i)\right)\boldsymbol{x}(i)+k_{ffSDRE}\left(\boldsymbol{x}(i)\right)w(i) \tag{29}$$

where $k_{ffSDRE}\left(\boldsymbol{x}(i)\right)$ is the state-dependent feedforward gain computed in every i-th time step as:

$$k_{ffSDRE}\left(\boldsymbol{x}(i)\right)=\frac{1}{\boldsymbol{c}_1\left(\boldsymbol{I}-\left(\boldsymbol{F}_i-\boldsymbol{g}_i\boldsymbol{k}_{SDRE}\left(\boldsymbol{x}(i)\right)\right)\right)^{-1}\boldsymbol{g}_i} \tag{30}$$

The *IPMaC* block library provides program support for the SDRE method in form of a function block, *State-Dependent Riccati Equation Controller,* included into the sublibrary *Inverted Pendula Control*. The stepwise computation of the optimal control input was implemented in form of interpreted MATLAB functions, called from *Simulink* at every time step, which ensures the required fast online computation of the control law.

IV. Case Study – Rotary Single Inverted Pendulum System

The performance of the described nonlinear control design method was verified using the simulation model of a *rotary single (Furuta) inverted pendulum system* [5][8]. This example of an underactuated IPS is composed of a rigid, homogenous pendulum rod attached to an arm rotating in a horizontal plane, where the only input (torque $M(t)$ applied on the arm) actuates both DoFs of the system: arm angle $\theta_0(t)$ [rad] and pendulum angle $\theta_1(t)$ [rad]. The mathematical model of the system consists of two second-order nonlinear differential equations which respectively describe the rotary arm and the pendulum. The standard minimal form of the model generated by the *Derivator* is:

$$\begin{aligned}&\begin{pmatrix}J_0+m_1l_0^2+\frac{1}{4}m_1l_1^2\sin^2\theta_1(t) & \frac{1}{2}m_1l_0l_1\cos\theta_1(t)\\ \frac{1}{2}m_1l_0l_1\cos\theta_1(t) & J_1\end{pmatrix}\begin{pmatrix}\ddot{\theta}_0(t)\\ \ddot{\theta}_1(t)\end{pmatrix}+\\ &+\begin{pmatrix}\delta_0+\frac{1}{4}m_1l_1^2\dot{\theta}_1(t)\sin 2\theta_1(t) & -\frac{1}{2}m_1l_0l_1\dot{\theta}_1(t)\sin\theta_1(t)\\ -\frac{1}{8}m_1l_1^2\dot{\theta}_0(t)\sin 2\theta_1(t) & \delta_1\end{pmatrix}\begin{pmatrix}\dot{\theta}_0(t)\\ \dot{\theta}_1(t)\end{pmatrix}+\\ &+\begin{pmatrix}0\\ -\frac{1}{2}m_1gl_1\sin\theta_1(t)\end{pmatrix}=\begin{pmatrix}M(t)\\ 0\end{pmatrix}\end{aligned} \tag{31}$$

where m_0, m_1 stand for the masses of the arm and the pendulum, l_0, l_1 are their respective lengths, δ_0, δ_1 are the damping constants in the joints of the arm and pendulum, $J_0 = \frac{1}{3} m_0 l_0^2$ and $J_1 = \frac{1}{3} m_1 l_1^2$ are the moments of inertia of the arm and pendulum with respect to their pivot points and $M(t)$ is the input torque applied upon the rotary arm [8].

The effectiveness of the SDRE-based nonlinear state-feedback controller was compared with that of the LQR controller based on the system's linearized model in two simulation experiments. The numeric parameters of the simulated rotary single inverted pendulum model were selected to be $m_0 = 0.5kg$, $m_1 = 0.275kg$, $l_0 = 0.6m$, $l_1 = 0.5m$, $\delta_0 = 0.3kgs^{-1}$, $\delta_1 = 0.011458kgm^2s^{-1}$ in all cases.

As the control objective in the first experiment, the pendulum had to be brought into the upright equilibrium after a nonzero initial deflection, and the arm needed to be stabilized at the origin. Fig. 2 depicts the effect of both considered controllers on the time behavior of the controlled system. The initial pendulum deflection was consecutively set to 15, 25 and 35 degrees away from the upright position and the weight matrices which occur in both the discrete-time algebraic Riccati equation and the state-dependent Riccati equation were specified to be $\boldsymbol{Q} = diag(500 \quad 0 \quad 20 \quad 0)$, $r = 1$.

It is obvious that *low values of initial deflection* cause slight to no difference between the performance of the SDRE controller and the constant discrete-time LQR controller since the pseudo-linear matrices are close to equal to the linearized matrices in the close vicinity of the equilibrium. For *higher initial deflections*, though, the SDRE controller yields results which are superior to those obtained by the discrete-time LQR controller in terms of both arm and pendulum overshoot. Further increase in the initial deflection proves that while the LQR controller is only effective if the full state of the system is sufficiently close to the equilibrium point, the nonlinear SDRE controller succeeds in stabilizing the system in a much larger region of recovery.

In the second experiment, the initial conditions were set to $\theta_0(t) = \theta_1(t) = 0$ and the control objective was to keep the pendulum upright while the arm rotates for a total of 155 degrees, stopping at predefined positions to stabilize before returning to its initial position. Two sets of weight matrices for the discrete-time infinite horizon cost function were selected in both cases: $\boldsymbol{Q}_1 = diag(500 \quad 0 \quad 20 \quad 0)$, $r = 1$ (emphasis on arm angle) and $\boldsymbol{Q}_2 = diag(35 \quad 200 \quad 20 \quad 0)$, $r = 1$ (emphasis on pendulum angle). Simulation results which illustrate the performance of both considered controllers with respect to selected weight matrices are depicted in Fig. 3.

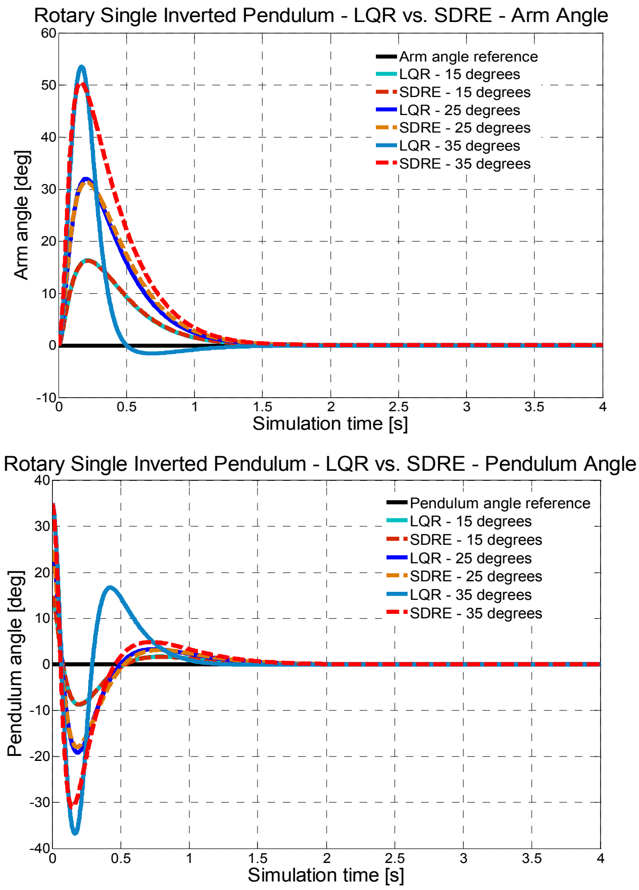

Figure 2. Rotary single inverted pendulum – comparison of the performance by LQR and SDRE in initial deflection control

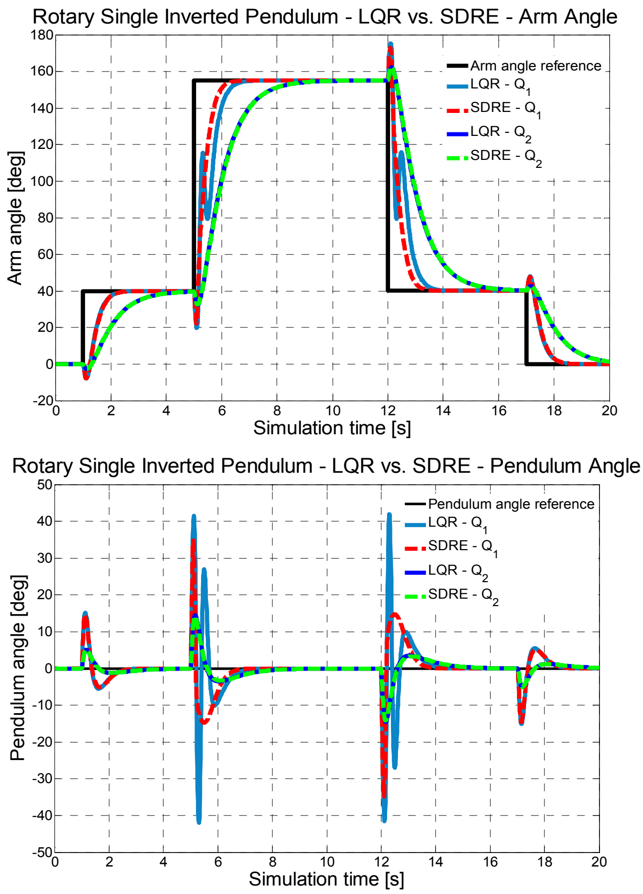

Figure 3. Rotary single inverted pendulum – comparison of the performance by LQR and SDRE in reference command tracking

Regardless of which control design technique is used, the *quick rise time of the arm* and the *low overshoot of the pendulum* show up as conflicting requirements which cannot be satisfied simultaneously. Any successful tuning of weight matrices must therefore result in a reasonable compromise between the two. Analogically to the previous experiment, the response of the discrete-time LQR controller closely follows that of the SDRE controller if a sufficiently low step change has been applied to the arm reference value. Greater differences to the reference value cause gradual deterioration of the discrete-time LQR controller performance, whereas that of the SDRE-based controller retains its superior quality even for high changes in reference value.

V. Conclusion

The purpose of this paper was to provide a practically-oriented overview of the nonlinear control design approach based on the state-dependent Riccati equation (SDRE). As a systematic and flexible way of designing nonlinear feedback controllers, the SDRE technique approximates the solution of the infinite horizon optimal control and serves as a nonlinear counterpart to the LQR-based control design. The individual steps of the SDRE control design were illustrated on an important representative of nonlinear mechanical underactuated systems: the generalized (n-link) system of classical and rotary inverted pendula.

The performance of the controller designed via the SDRE control technique was verified using the simulation model of a rotary single inverted pendulum system, and the results were compared to those yielded by the standard LQR control design, which relies on a linearized model around the equilibrium. Although the accuracy of a linear approximation is generally sufficient for control algorithm design for underactuated systems, it was nevertheless shown that the SDRE-based controller, which preserves the original nonlinear system dynamics, gradually outperforms the conventional discrete-time LQR algorithm whenever the distance from the equilibrium point increases.

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