# Matlab-Based Tools for Analysis and Control of Inverted Pendula Systems

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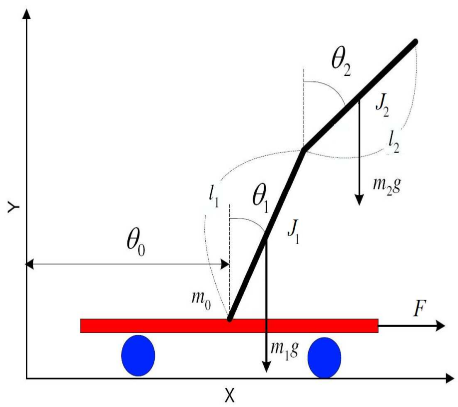
***Abstract* —The aim of this paper is to provide an original approach to the analysis and control of inverted pendula systems which employs custom blocks and demo schemes of a Simulink block library designed by the authors. The capabilities of the library are enhanced by software tools which provide a user-friendly graphical interface to modeling and linearization. The tools are next shown in action as classical double and rotary single pendulum systems are analyzed, modeled and successfully stabilized in the unstable inverted position of the pendula.**

***Keywords* — system of inverted pendula, state-space control, MATLAB/Simulink block library.**

## I. Introduction

Inverted pendula systems represent a significant group of mechanical systems used in control education with a number of practical applications. It is evident that the stabilization of a walking human or robot, a launching missile or the vertical movement of a shoulder or arm can all be modeled by some kind of an inverted pendula system. The diversity of modeled systems is reflected in the variety of available inverted pendula models. These may differ by:

- *the type of actuating mechanism* – the system base is moving either in a single axis (*classical pendulum system*) or in a plane (*rotary pendulum system*)
- *the number of pendulum links attached to the mechanism* – single and double pendulum systems are common control plants; triple and quadruple pendulum are rare but controllable
- *the distribution of mass within the pendulum rod* – the pendulum links are either homogenous rods with the mass concentrated in the center of gravity; or the rod is considered massless and the mass is concentrated in the load at its end.



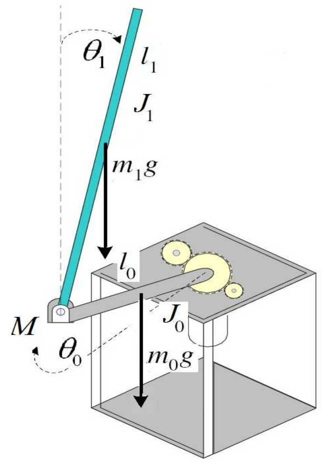




Fig. 1 a) Classical double inverted pendulum system - scheme
1 b) Rotary single inverted pendulum system – scheme

The most standard representative of the family of inverted pendula systems, the classical single inverted pendulum system, was thoroughly analyzed in [1][2] together with suitable control algorithms and therefore will not be included in this paper. This paper will instead focus on the analysis and control of the more challenging systems, i.e.:

- double (two-link) classical inverted pendulum system (Fig. 1a)
- single (one-link) rotary inverted pendulum system (Fig. 1b)

## II. A LIBRARY OF MODELS

The nonlinear *mechanic system of n inverted pendula* is basically a set of $n$>1 pendulum links attached to a stable base which may be moveable in one axis (cart) or in a horizontal plane (rotary arm). It is a typical example of an underactuated system since the number of actuators is lower than the number of system links: the only input (the force acting upon the cart/ the momentum applied on the rotary arm) is used to control the $n$+1 outputs of the system: cart position $[m]$ or arm angle $[rad]$, and pendula angles $[rad]$.

Since 2009, a thematic Simulink block library *Inverted Pendula Modeling and Control* (*IPMaC*) has been developed. The purpose of the block library is to provide software support for analysis, simulation and control of inverted pendula systems using custom-designed blocks and „demo“ simulation schemes which illustrate the way the blocks may be interconnected to solve various analysis- and control-related problems. The installation process and sublibrary structure of the *IPMaC* was described in detail in [1].

### *A. Derivation of Motion Equations*

An integral part of the *IPMaC*, the *Inverted Pendula Model Equation Derivator*, is a MATLAB GUI tool which generates the motion equations for a user-chosen type of inverted pendula system. Such automatic approach has a number of advantages: it yields a particularly precise approximation of the real system's dynamics and eliminates any factual or numeric errors which could arise during manual mathematical modeling. Fig. 2 shows a preview of the *Derivator* output for the single rotary inverted pendulum system.

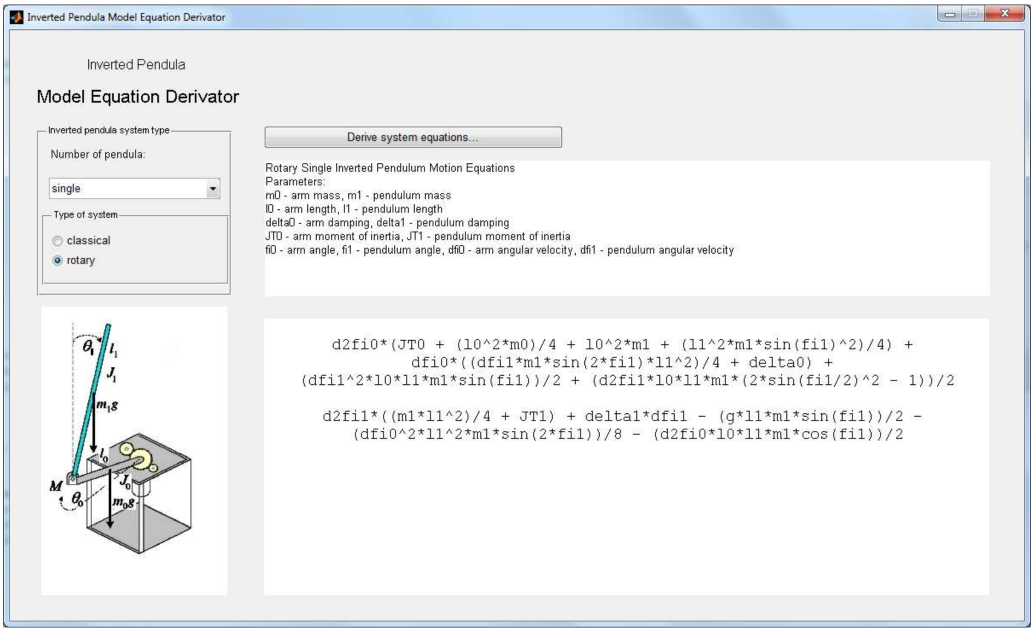



Fig. 2 *Inverted Pendula Model Equation Derivator* GUI tool in use:
rotary pendulum model derivation

The core of the *Derivator* tool is represented by MATLAB functions that use the *Symbolic Math Toolbox* to implement general procedures that derive the motion equations for a classical or rotary inverted pendula system. If we represent the system's outputs as a vector of generalized coordinates [2]:

$$\boldsymbol{\theta}(t) = \begin{pmatrix} \theta_0(t) & \theta_1(t) & \dots & \theta_n(t) \end{pmatrix}^T \tag{1}$$

then the system can be mathematically described by the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L(t)}{\partial \dot{\boldsymbol{\theta}}(t)}\right) - \frac{\partial L(t)}{\partial \boldsymbol{\theta}(t)} + \frac{\partial D(t)}{\partial \dot{\boldsymbol{\theta}}(t)} = \boldsymbol{Q}^*(t) \tag{2}$$

where $L(t)$ (*Lagrange function*) is defined as the difference between the system's kinetic and potential energy, $D(t)$ (*Rayleigh, dissipation function*) describes the viscous (friction) forces and $\boldsymbol{Q}^*(t)$ is the vector of *generalized external forces* acting upon the system. The process of derivation of the motion equations to describe any kind of inverted pendula system is hence transformed into the determination of kinetic, potential and dissipation energies related to the base and all pendula. Using well-known physical formulae, general relations that describe the energetic balances of the base and *i*-th pendulum in an inverted pendula system were derived. These can be found in [2] for the system of inverted pendula on a cart and derived analogously for the rotary pendulum system. The whole derivation process can be tracked in the command window (see the preview in [1]).

Using the *Derivator* tool, mathematical models of both the classical double and rotary single pendulum system were generated. The motion equations were rewritten into the standard (minimal ODE – ordinary differential equation) form:

$$\boldsymbol{M}\left(\boldsymbol{\theta}(t)\right)\ddot{\boldsymbol{\theta}}(t)+\boldsymbol{N}\left(\boldsymbol{\theta}(t),\dot{\boldsymbol{\theta}}(t)\right)\dot{\boldsymbol{\theta}}(t)+\boldsymbol{P}\left(\boldsymbol{\theta}(t)\right)=\boldsymbol{V}(t) \tag{3}$$

which provides the only way to express this kind of system in the nonlinear state-space form of

$$\begin{aligned}\dot{\boldsymbol{x}}(t)&=\boldsymbol{f}\left(\boldsymbol{x}(t),u(t),t\right)\\ \boldsymbol{y}(t)&=\boldsymbol{g}\left(\boldsymbol{x}(t),u(t),t\right)\end{aligned} \tag{4}$$

by defining the state vector as $\boldsymbol{x}(t)=\begin{pmatrix}\boldsymbol{\theta}(t) & \dot{\boldsymbol{\theta}}(t)\end{pmatrix}^T$ and isolating the second derivative $\ddot{\boldsymbol{\theta}}(t)$ from (3).

### B. Selected inverted pendula models – model analysis

The *classical double inverted pendulum system* (Fig. 1a) is composed of a pair of rigid rods which are interconnected in a joint and one of these is attached to a cart. The mathematical model of the system has the form of three rather complex second-order nonlinear differential equations which describe the dynamic behavior of the cart and both pendulum links:

$$\begin{pmatrix} m_0+m_1+m_2 & \left(\frac{1}{2}m_1l_1+m_2l_1\right)\cos\theta_1(t) & \frac{1}{2}m_2l_2\cos\theta_2(t) \\ \left(\frac{1}{2}m_1l_1+m_2l_1\right)\cos\theta_1(t) & J_1+m_2l_1^2 & \frac{1}{2}m_2l_1l_2\cos\left(\theta_1(t)-\theta_2(t)\right) \\ \frac{1}{2}m_2l_2\cos\theta_2(t) & \frac{1}{2}m_2l_1l_2\cos\left(\theta_1(t)-\theta_2(t)\right) & J_2 \end{pmatrix}\begin{pmatrix}\ddot{\theta}_0(t)\\ \ddot{\theta}_1(t)\\ \ddot{\theta}_2(t)\end{pmatrix}+$$

$$+\begin{pmatrix} \delta_0 & -\left(\frac{1}{2}m_1l_1+m_2l_1\right)\dot{\theta}_1(t)\sin\theta_1(t) & \frac{1}{2}m_2l_2\cos\theta_2(t) \\ \frac{1}{2}m_1l_1\cos\theta_1(t) & \delta_1+\delta_2 & -\delta_2-\frac{1}{2}m_2l_1l_2\dot{\theta}_2(t)\sin\left(\theta_1(t)-\theta_2(t)\right) \\ \frac{1}{2}m_2l_2\cos\theta_2(t) & -\delta_2-\frac{1}{2}m_2l_1l_2\dot{\theta}_2(t)\sin\left(\theta_1(t)-\theta_2(t)\right) & \delta_2 \end{pmatrix}\begin{pmatrix}\dot{\theta}_0(t)\\ \dot{\theta}_1(t)\\ \dot{\theta}_2(t)\end{pmatrix}+$$

$$+\begin{pmatrix} 0 \\ -\left(\frac{1}{2}m_1+m_2\right)gl_1\sin\theta_1(t) \\ -\frac{1}{2}m_2gl_2\sin\theta_2(t)\end{pmatrix}=\begin{pmatrix}F(t)\\ 0\\ 0\end{pmatrix} \tag{5}$$

where $m_0$ is the cart mass, $m_1$, $m_2$ are the pendula masses, $l_1$, $l_2$ are the pendula lengths, $\delta_0$ is the friction coefficient of the cart, $\delta_1$, $\delta_2$ are the damping constants in the joints of the pendula, $J_1=\frac{1}{3}m_1l_1^2$, $J_2=\frac{1}{3}m_2l_2^2$ are the moment of inertia of the pendula with respect to the pivot points and $F(t)$ is the force induced on the cart.

The *rotary single inverted pendulum system* (Fig. 1b) consists of a pendulum rod attached to an arm rotating in a horizontal plane. Once again, the mathematical model of the system, as it was generated by the *Derivator*, is composed of two second-order nonlinear differential equations which respectively correspond to the rotary arm and the pendulum:

$$\begin{pmatrix} J_0 + m_1 l_0^2 + \frac{1}{4} m_1 l_1^2 \sin^2 \theta_1(t) & -\frac{1}{2} m_1 l_0 l_1 \cos\theta_1(t) \\ -\frac{1}{2} m_1 l_0 l_1 \cos\theta_1(t) & J_1 \end{pmatrix} \begin{pmatrix} \ddot{\theta}_0(t) \\ \ddot{\theta}_1(t) \end{pmatrix} + \\ + \begin{pmatrix} \delta_0 & -\frac{1}{2} m_1 l_0 l_1 \dot{\theta}_1(t) \sin\theta_1(t) \\ -\frac{1}{8} m_1 l_1^2 \dot{\theta}_0(t) \sin 2\theta_1(t) & \delta_1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_0(t) \\ \dot{\theta}_1(t) \end{pmatrix} + \\ + \begin{pmatrix} 0 \\ -\frac{1}{2} m_1 g l_1 \sin\theta_1(t) \end{pmatrix} = \begin{pmatrix} M(t) \\ 0 \end{pmatrix} \tag{6}$$

where $m_0$, $m_1$ stand for the masses of the arm and the pendulum, $l_0$, $l_1$ are their respective lengths, $\delta_0$, $\delta_1$ are the damping constants in the joints of the arm and pendulum, $J_0 = \frac{1}{3} m_0 l_0^2$ and $J_1 = \frac{1}{3} m_1 l_1^2$ are the moments of inertia of the arm and the pendulum with respect to their pivot points and $M(t)$ is the input momentum applied on the arm.

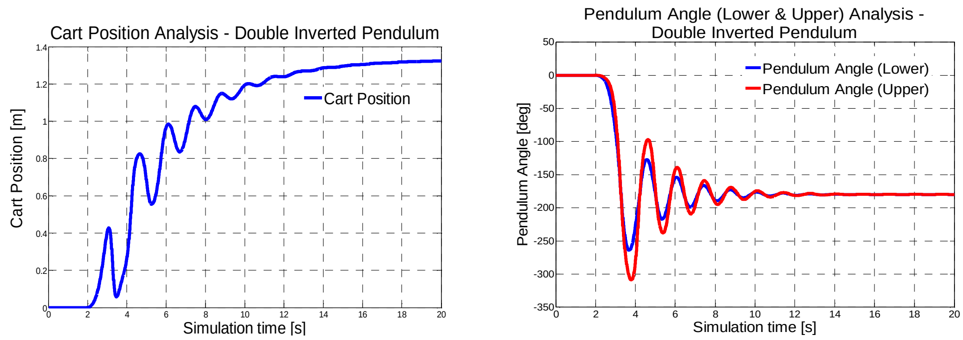



Fig. 3 Classical double inverted pendulum system time behavior – cart position and pendula angles

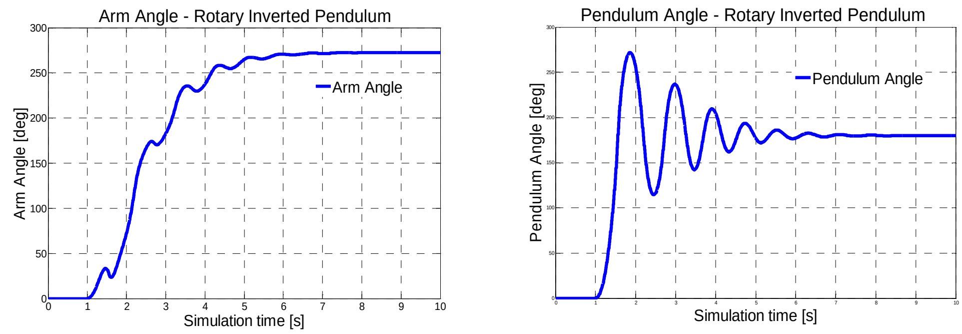



Fig. 4 Rotary single inverted pendulum system time behavior – arm and pendulum angles

Both models were included in the *IPMaC* block library in form of atomic Simulink blocks with a dynamic mask which supports useful features such as editable parameter constants and initial conditions as well as an adjustable number of input and output ports [1]. The obvious complexity of model equations is the price paid for a particularly accurate simulation model. To determine the behavior of the models in response to an impulse signal, simulation experiments were performed, to satisfactory results (Fig. 3, Fig. 4). Long-observed empirical findings about pendula behavior are confirmed: each pendulum of the system passes through oscillatory transient state until the system reaches the stable equilibrium point with all pendula pointing downward. The backward impact of the pendulum/pendula on the base (cart/arm), which increases with the weight of the load, is also visible. The models can therefore be considered accurate enough to serve as a reliable testbed for control algorithms.

## III. Control Algorithm Blocks

Controllability properties of inverted pendula systems were verified using both the double and rotary inverted pendulum system [3][4][5]. In order to meet the standard control objective in pendula systems, i.e. to stabilize all pendulum links in the upright, unstable position, *linear* methods of synthesis were used. As a result, linear approximation of the originally nonlinear inverted pendulum systems is required. The process can be considerably sped up with help of another GUI tool from the *IPMaC*: the *Inverted Pendula Model Linearizator & Discretizer*. In case the system type, model parameters and equilibrium point have been provided by the user, the tool generates the numeric state-space matrices of a linearized system using the standard (Taylor) series expansion around a given equilibrium point (Fig. 5)[3]. The discretized state-space matrices of the system, necessary for discrete state-space control design, are also returned if the sample time constant has been provided.

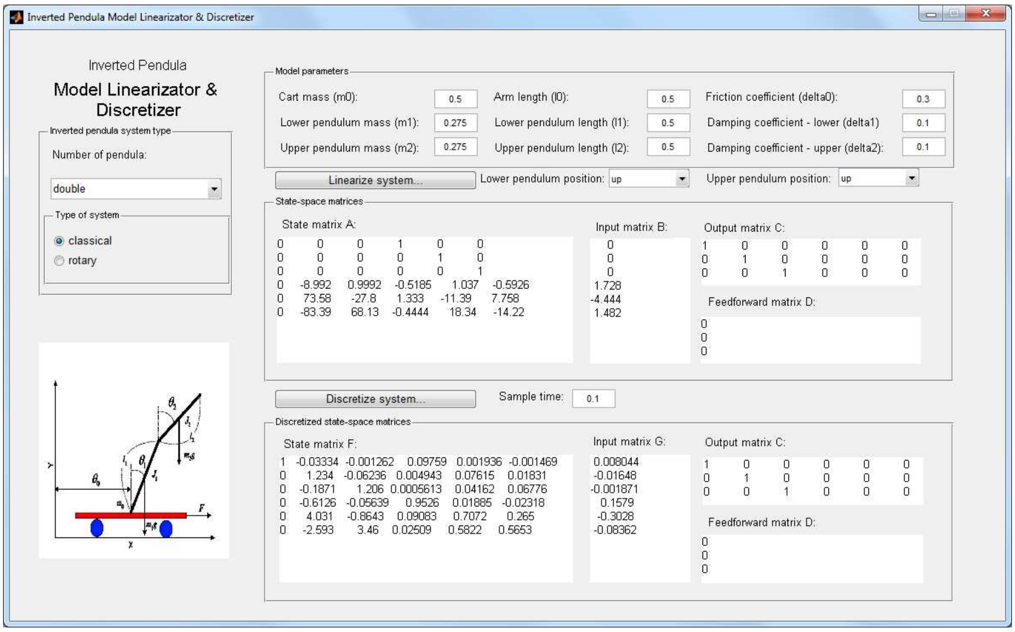



Fig. 5 *Inverted Pendula Model Linearizator & Discretizer* GUI tool in use:
linearized and discretized state-space matrices of the double inverted pendulum system

All inverted pendula systems included in the *IPMaC* were modeled in a way which defines the "all upright" equilibrium as $\boldsymbol{x}(t) = \boldsymbol{x}_S = \boldsymbol{0}^T$. If $u(t) = u_S = 0$, the state-space description of the continuous linearized system is given as

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t) \\ y(t) &= \boldsymbol{C}\boldsymbol{x}(t) + du(t) \end{aligned}, \tag{7}$$

in case $\boldsymbol{A}$, $\boldsymbol{b}$, $\boldsymbol{C}$, $d$ are the numeric continuous state-space matrices generated by the *Linearizator & Discretizer* tool.

To provide program support for inverted pendula control, linear state-space algorithms were implemented into Simulink and encapsulated into dynamic-masked control blocks. Most importantly, the *State Space Controller* block evaluates the relation

$$u(t) = u_f(t) + u_v(t) + d_u(t) = -\boldsymbol{k}\boldsymbol{x}(t) + \boldsymbol{k}_v w(t) + d_u(t), \tag{8}$$

where $\boldsymbol{k}$ is the *feedback gain* which brings the system's state vector to the origin of the state space [3][4], $\boldsymbol{k}_v$ represents the *setpoint gain* which needs to be applied if a nonzero required value is specified and $d_u(t)$ is the unmeasured disturbance. To match an additional control objective (initial deflection, compensation of disturbance signal, tracking a reference position of the cart or a combination of the three), the block's appearance may be adjusted by optional enabling or disabling of the nonzero setpoint input $\boldsymbol{w}(t)$ and the disturbance input $d_u(t)$.

The *State Estimator* block implements the Luenberger state estimator:

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{A}\hat{\boldsymbol{x}}(t) + \boldsymbol{b}u(t) + \boldsymbol{L}\left(\boldsymbol{y}(t) - \boldsymbol{C}\hat{\boldsymbol{x}}(t)\right) \tag{9}$$

where $\boldsymbol{L}$ is the estimator gain matrix and $\hat{\boldsymbol{x}}(t)$ is the estimated state vector ([2][6]).

The *Demo Simulations* section of the *IPMaC* documents several simulation experiments which were performed to prove that both the LQR-designed controller and standard pole-placement are able to meet the required control objectives for both systems in question. Fig. 6 and Fig. 7 show the simulation results in case the control objective was to maintain the desired cart position/arm angle while keeping the pendulum/pendula upright. Measurement limitations were simulated and an estimator block was included in both schemes to provide the controller block with a complete state-space vector (see [2] for the structure of control loop)

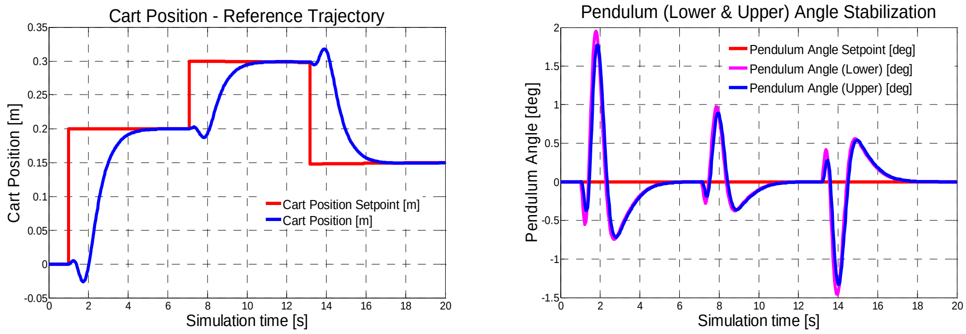



Fig. 6 Classical double inverted pendulum: simulation results for pole-placement control, estimator included

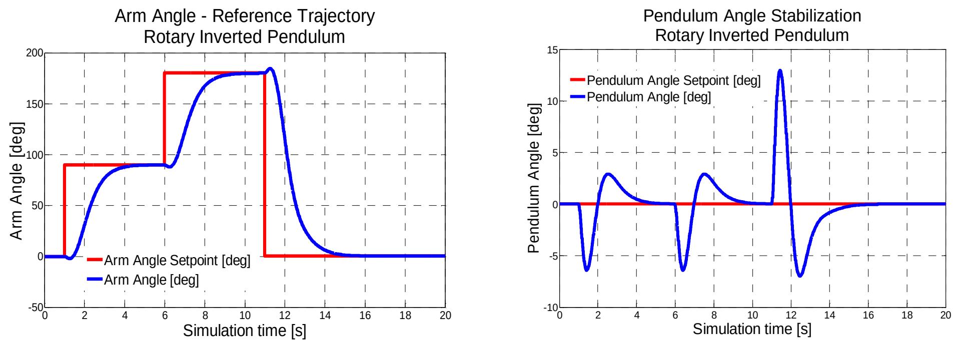



Fig. 7 Rotary single inverted pendulum: simulation results for LQR control, estimator included; note that the arm is supposed to rotate for exactly a half-circle before returning to its initial position

## IV. Conclusion

The purpose of this paper was to propose an original conception of solving the task of modeling and control of the inverted pendula dynamical systems. Using the custom-designed Simulink block library *Inverted Pendula Modeling and Control* and custom program tools with graphical user interface, classical double and rotary single pendulum systems were analyzed and modeled. Algorithms of linear state-space control that stabilize the pendula in the inverted position were incorporated into the demo simulation archive of the library.

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