OPTIMIZATION TOOLBOX USAGE IN SOLVING VECTOR OPTIMIZATION TASKS

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Abstract

Aim of this paper is to describe functional and application possibilities of complex application created for resolving vector optimization tasks with two objectives. MATLAB, its Optimization Toolbox and Guide features were used to design application VOPTIM, which provides users with intuitive interface developed for simple using. Main feature of this application is ability of choosing and comparing various vector optimization methods to find out the best one for solving particular problem. Objectivity, reliability and mathematical basis of these methods are the warranty of getting satisfying results, which can be applicable in production, economy, shipping or other industrial sectors.

1   Optimization introduction

Optimization can be considered as choosing the best option from wide spectrum of alternatives. In our everyday life we are trying to make our work done spending as little time or as little energy as possible. Optimization methods are used to solve this problem most frequently, mostly because of their mathematical basis, which guarantees objectivity and accuracy of optimization process.

Optimization methods are used in many different areas. In economical sphere there is a main goal of maximizing profit or minimizing costs; in travelling it is reducing cost and time of travel. Other example is customer in the shop who is trying to maximize his satisfaction, but in the same time he wants to spend minimum money to do so. Publications dealing with optimization problems and its solving are [1] and [6].

Optimization process is disposing with apparatus to choose the best possible solution for particular task. In order to make final decision responsibly and competently, it is necessary to:

- create mathematical model which precisely describes the situation. Model represents mathematical description of real system. It has to contain quantifiable parameters, which are used to evaluate the success rate of optimization process (profit, costs etc.).Model can also contents some constrains (f.e. maximal amount of invested money);
- find the solution of particular optimization problem using proper algorithm, because there is no universal algorithm, which can be used to solve all optimization tasks (tasks solved by methods of linear and quadratic programming are described in this paper);
- verify and analyze found solution (if it is the real solution of optimization task). It is also necessary to interpret the solution correctly.[6]

Optimization problems can be divided into many different groups. Choosing the best optimization method is affected by form of objective function as well as by number of objective functions.

Application VOPTIM was created in MATLAB, using its Optimization Toolbox, to solve vector optimization tasks. It is complex application designed to solve vector optimization problems with two objective functions, one of which has to have linear form, other one might have quadratic as well as linear form. Some possibilities of solving these problems can be found in [1] or [5], but there was no complex application able to solve vector optimization tasks.

VOPTIM was designed to provide users with simple and intuitive interface for solving specific portfolio of optimization problems. Functionality of this application is displayed by solving 2 model vector optimization problems:

- production optimization problem,

- optimal investment strategy problem.

2 VOPTIM application for solving vector optimization tasks

Application VOPTIM was designed in MATLAB in order to deal with vector optimization tasks. General optimization problem can be defined as minimization (maximization) of objective function

$$f = \{x_1, x_2, \dots, x_n\} \tag{1}$$

with respecting all constrains

$$g_i = \{x_1, x_2, \dots, x_n\}, \qquad for\ i = 1,2, \dots, n$$
$$x_j > 0, \qquad for\ j = 1,2, \dots, n. \tag{2}$$

Vector optimization is taking place, when we have to consider more than one objective function in finding result of optimization process. It is dealing with ways of optimizing the problem with multiple goals. It is used when it is necessary to accept more than one factor in finding the ideal solution of optimization task. To find proper vector optimization result, it is necessary to solve partial optimization problems first. For this purpose, methods of mathematical programming are used. Depending on the type of objective function, these methods can be divided into:
- linear programming methods,
- non linear programming methods,
- integer programming methods
- parameter programming methods,
- stochastic programming methods.

2.1 Solving partial optimization problems

In following parts of the paper there will be a more detail description of linear and quadratic programming methods used to solve optimization tasks. Focus on these methods is connected with specialization for these methods in VOPTIM application.

Linear programming task description

Linear programming is part of mathematical programming dealing with finding optimal solutions of optimization tasks.

Most of optimization tasks consists of two parts:
- main goal – defined by objective function,
- constrains .

Both of these parts are described by linear functions with multiple variables. Generally the linear programming problem can be described with following mathematical model:

$$z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n \ \to opt(min, max) \tag{3}$$

with respecting constrains

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n >=< b_1,$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n >=< b_2,$$
$$\dots \dots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n >=< b_n. \tag{4}$$

Other way to define the task is to use the matrix form:

$$z = c^T x \to opt(min, max)$$
$$Ax >=< b, \tag{5}$$

in which c is vertical vector of objective function coefficients, x is vertical vector of task variables , A is matrix describing left sides of constrains a b is vertical vector describing right sides of constrains.[4]

Quadratic programming task description

Goal of quadratic programming task is to optimize quadratic objective function with linear constrains. To describe general form of quadratic programming task, there is a need to mention some terms inevitable for closer definition of this form.

There is a symmetrical and positively semidefinite matrix $C \in \mathbb{R}^n$ and there is $d, a_i \in \mathbb{R}^n, b_i \in \mathbb{R}$, $i = 1, 2, ..., m$. Let's label:

$$P = \left\{ x \in \mathbb{R}^n, x_1, ..., x_s \geq 0, s \in \{0, ..., n\} \right\}$$

$$X = \{ x \in P \langle a_i, x \rangle \leq b_i, i = 1, ..., k; \langle a_i, x \rangle = b_i, i = k+1, ..., m \} \tag{6}$$

After this definition we can define general form of quadratic programming task as

$$f(x) = \frac{1}{2} \langle Cx, x \rangle + \langle d, x \rangle \rightarrow min, x \in X. \tag{7}$$

We will label $A = [a_1, ..., a_m]^T$, where $a_1, ..., a_n \in \mathbb{R}^n$ are rows of matrix A a $b = [b_1, ..., b_m]^T$. With closer knowledge it is possible to define Lagrange function of quadratic programming task:

$$L = \frac{1}{2} \langle Cx, x \rangle + \langle A^T y + d, x \rangle - \langle y, b \rangle, \tag{8}$$

and so

$$L'_x(x, y) = Cx + A^T y + d. \tag{9}$$

It is important to mention, that $y \in Q = \{y = (y_1, ..., y_m), y_1, ... y_k \geq 0\}$.

Solution of quadratic programming task is x^* just when there is $y^* \in Q$ respecting so called Kuhn – Tucker conditions:

$$\langle Cx^* + A^T y^* + d, x - x^* \rangle \geq 0, \forall x \in P$$

$$y_i^*(Ax^* - b)_i = 0, i = 1, ..., m. \text{ [5]} \tag{10}$$

2.2 VOPTIM application description

Application VOPTIM is able to solve vector optimization problems with two objective functions. One of them must have linear form, second may be either quadratic or also linear. All constrains have to be defined by linear equalities or inequalities. VOPTIM has its unique user interface created in `guide` MATLAB tool. It is used to simple and intuitive entering inputs as well as to watching and analyzing computational process outputs. Built-in functions of MATLAB Optimization Toolbox (`linprog, quadprog`) were used to solve partial linear or quadratic optimization problems. Also script for solving specific vector optimization problems was created as main product of this application. [2],[7].

Functionality of VOPTIM applications is described using 2 model vector optimization problems. First one is production optimization problem defined by 2 linear objective functions, other one is dealing with ideal investment strategy, where one linear and one quadratic objective function are taking place.

Application VOPTIM is able to solve two types of vector optimization (VO) tasks:

- VO problem with 2 objective functions, both of them have linear form
- VO problem with 2 objective functions, one of them has linear form and one has quadratic form

First menu after opening the application provides user with choosing a type of vector optimization problem he wants to solve.
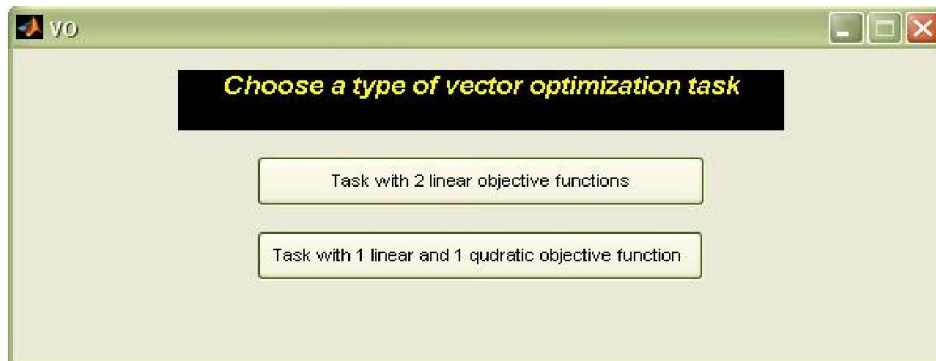
Figure 1: VOPTIM application start menu

After choosing type of vector optimization task, main window is displayed. In this window user is putting in input parameters, choosing method of vector optimization. After computational process he is able to watch and analyze results.
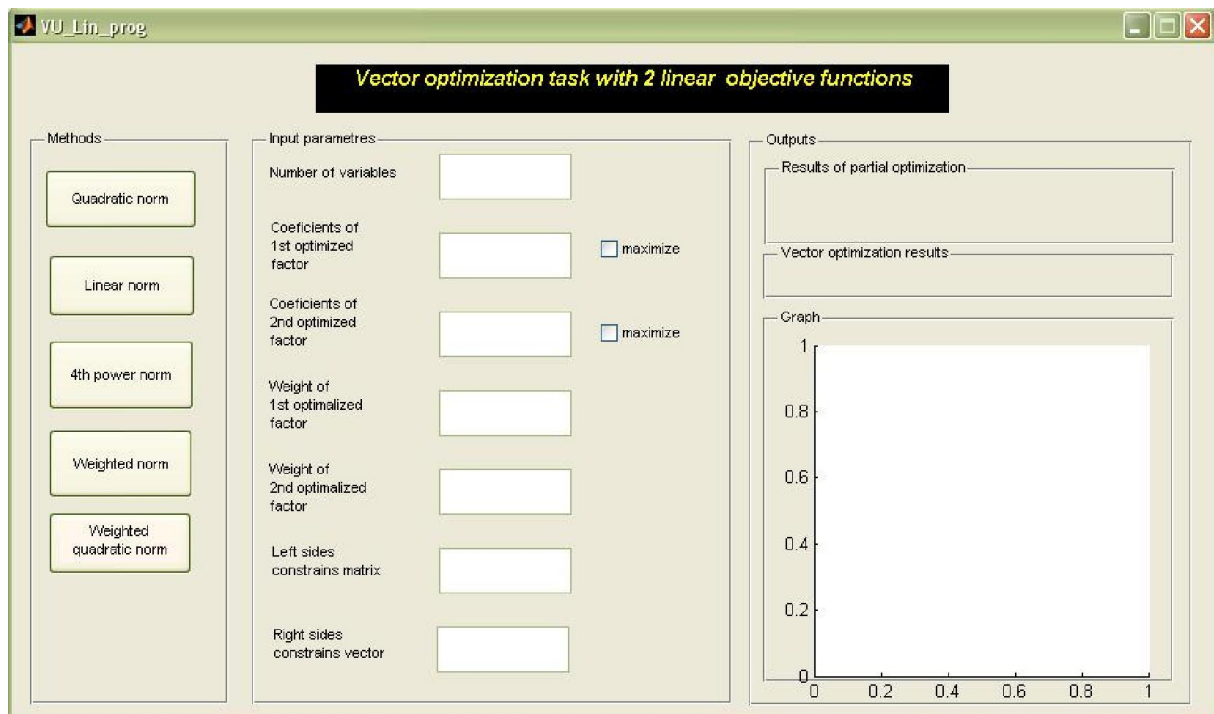

Figure 2: Main window of VO task with 2 linear objective functions

Main Window has 3 panels:

- Methods panel – panel used for choosing method of vector optimization,
- Input parameters panel – panel for putting in input parameters,
- Outputs panel – panel used for displaying outputs.[2]

2.2.1 Methods panel

This panel contains buttons of 5 methods used for solving vector optimization tasks. All methods have same input parameters, weighted methods are using also weights in computational process. VO methods used in this application are:

Quadratic norm

The most common criterion used in this norm is minimal sum of quadratic variance of objective functions $J_\alpha(x)$ for random $x \in \{X\}$ from objective function $J_\alpha(x_\alpha)$ for vector of ideal values of chosen criteria $x_\alpha = (x_{1\alpha}, x_{2\alpha}, \ldots, x_{n\alpha})$, $\alpha = 1, 2, \ldots, k$; Values of this norm are usually divided by optimal value. Reason for this is non-dimensional solution value. Calculation formula then looks like:

$$R(x) = \sum_{\alpha=1}^{k} \frac{(J_\alpha(x) - J_\alpha(x_\alpha))^2}{J_\alpha(x_\alpha)^2}$$

$$opt\, R(x) = \min_{x \in \{X\}} \sum_{\alpha=1}^{k} \frac{(J_\alpha(x) - J_\alpha(x_\alpha))^2}{J_\alpha(x_\alpha)^2}.$$

(11)

Linear norm

It represents minimal sum of linear variance of objective functions $(J_\alpha(x))$ from optimal values of objective functions $J_\alpha(x_\alpha)$. Optimal value of functions $R(x)$ can be calculated using formulas:

$$R(x) = \left| \sum_{\alpha=1}^{k} (J_\alpha(x) - J_\alpha(x_\alpha)) \right|,$$

$$opt\, R(x) = \min_{x \in \{X\}} \left(\sum_{\alpha=1}^{k} (J_\alpha(x) - J_\alpha(x_\alpha)) \right).$$

(12)

Generalized norm

Objective functions of this norm is

$$R_L(x) = \sum_{\alpha=1}^{k} \left\{ \left((J_\alpha(x) - J_\alpha(x_\alpha))^L \right) \right\}^{\frac{1}{L}} ; L \geq 1.$$

(13)

For $L = 1$ this functions corresponds to linear norm, for $L = 2$ it corresponds quadratic norm and for $L = \infty$ it is

$$R_\infty(x) = \max_{\beta} \{ (J_\alpha(x) - J_\alpha(x_\alpha); = 1, 2, \ldots, n \}.$$

(14)

In this application generalized norm is represented by 4th power norm, for L = 4.

Weighted norms

In these norms function $R(x)$ is represented by formula

$$R(x) = \sum_{\alpha=1}^{k} \lambda_\alpha * (J_\alpha(x) - J_\alpha(x_\alpha)).$$

(15)

Optimal solution is always represented by minimum value of function $R(x)$, because this norm is using variation from ideal values.

$$opt\, R(x) = \min_{x \in \{X\}} \left(\sum_{\alpha=1}^{k} \lambda_\alpha * (J_\alpha - J_\alpha(x_\alpha)) \right).$$

(16)

It is also possible to use quadratic variant of this formula (weighted quadratic norm). [1]

2.2.2   Input parameters panel

This panel is used for inputting all parameters necessary for start of computational process. There are strict rules for format of all input parameters. When there are some incorrectly written outputs, computational process can't start and user is informed, which parameter was not properly entered.

2.2.3 Outputs panel

After finishing computational process, all results are displayed in outputs panel. It has three parts, one of which contains outputs of partial optimization for all objective functions. Second one is displaying results of vector optimization and the last part of this panel is graph showing values of all variables for partial optimizations, as well as for vector optimization.

2.3 Algorithm solving vector optimization tasks

For simple and intuitive vector optimization tasks solving, algorithm using `linprog` and `quadprog` functions for dealing with partial optimization problems was created. Main scheme of processes created to solve specific vector optimization task is displayed in Figure 3.
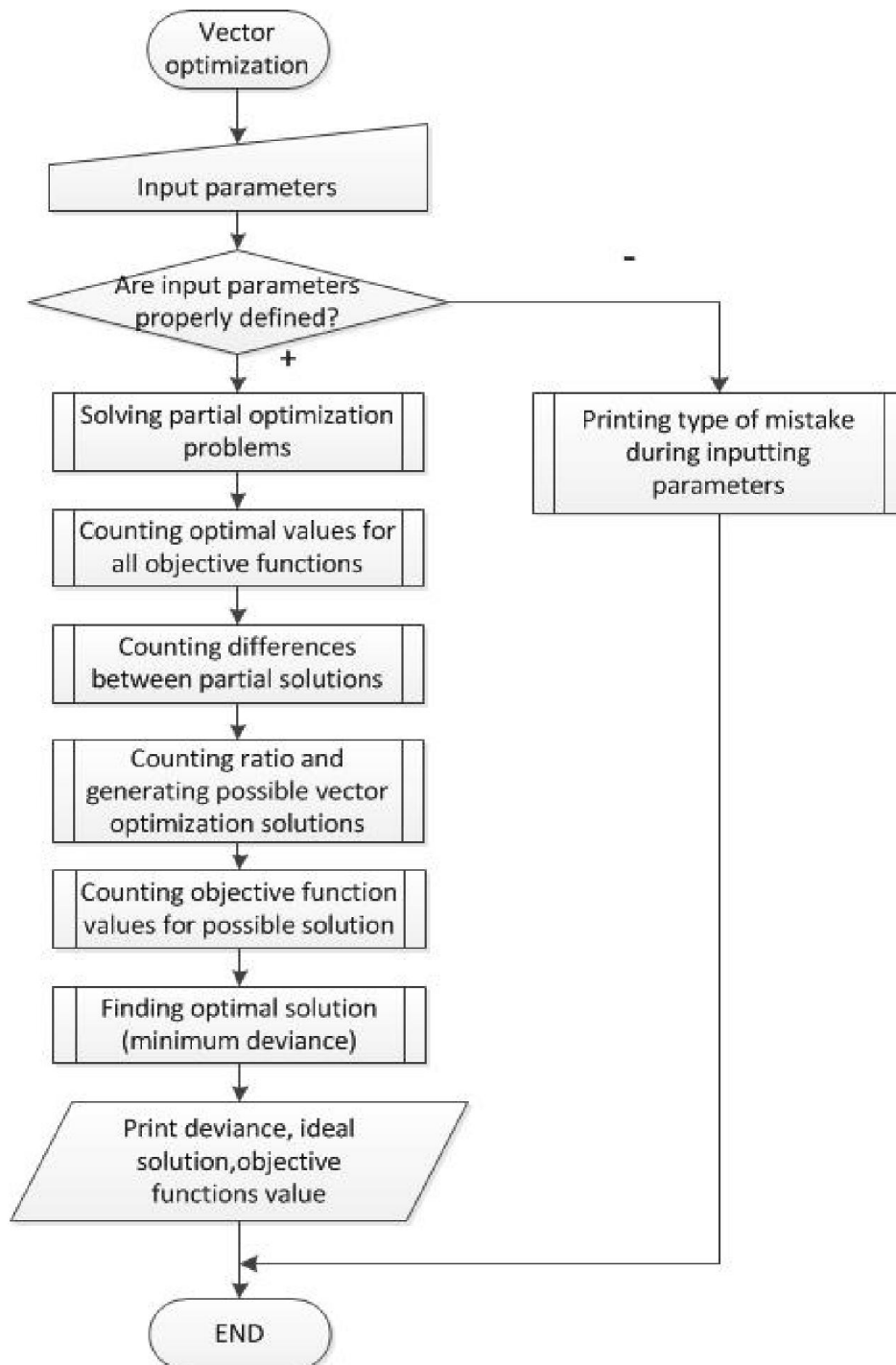
Figure 3: Vector optimization process algorithm

3 VOPTIM usage for solving production optimization problem

Model vector optimization problems are solved in this part of paper. Firstly, production optimization problem described by 2 linear objective functions is taking place. Other one is investors problem dealing with choosing ideal investment strategy. Objectives of this problem are described by one linear and one quadratic objective function.

3.1 Product optimization problem

Company is specialized on making 2 types of tools: hammers and screwdrivers. It uses two types of material for creating these tools: wood for handles and iron for effective part of tools. For one hammer, 80 units of iron and 60 units of wood is needed. For one screwdriver it is 30 iron units and 40 wood units. Company disposes with 2400 units of iron and same amount of wood units. Profit from one hammer is 3€, from one screwdriver it is 4€. Processing one hammer is saving 30 minutes, while processing one screwdriver is saving only 15 minutes. How many hammer and how many screwdrivers should company produce to maximize profit, as well as to maximize amount of saved time?

Problem Solving

Firstly it is necessary to make notation for this vector optimization tasks. This task has 2 objective functions (maximizing profit and amount of saved time), 2 variables (number of produced hammers and screwdrivers), and some linear constrains (amount of iron and wood owned by company). This vector optimization tasks could be defined as optimizing objective functions

$$f_1(x_1, x_2) = 3x_1 + 4x_2 \rightarrow \max$$

$$f_2(x_1, x_2) = 30x_1 + 15x_2 \rightarrow \max$$

with respecting constrains

$$80x_1 + 30x_2 \leq 2400$$

$$60x_1 + 40x_2 \leq 2400$$

$$x_1, x_2 \leq 0.$$

For finding ideal solution of partial linear programming problems, `linprog` function of MATLAB Optimization Toolbox is used. Theoretical background of linear programming method is described in chapter 2.1. In Table 1 are displayed results of partial linear optimization problems.

Table 1: RESULTS OF PARTIAL OPTIMIZATION PROBLEMS SOLVING

	NUMBER OF HAMMERS	*NUMBER OF SCREWDRIVERS*
Optimization for profit	0	60
Optimization for time saving	17	34

For solving this problem in VOPTIM application there is a need to insert input parameters correctly. After correct input and choosing vector optimization method computational process is taking place. Its results are displayed on Figure 4.
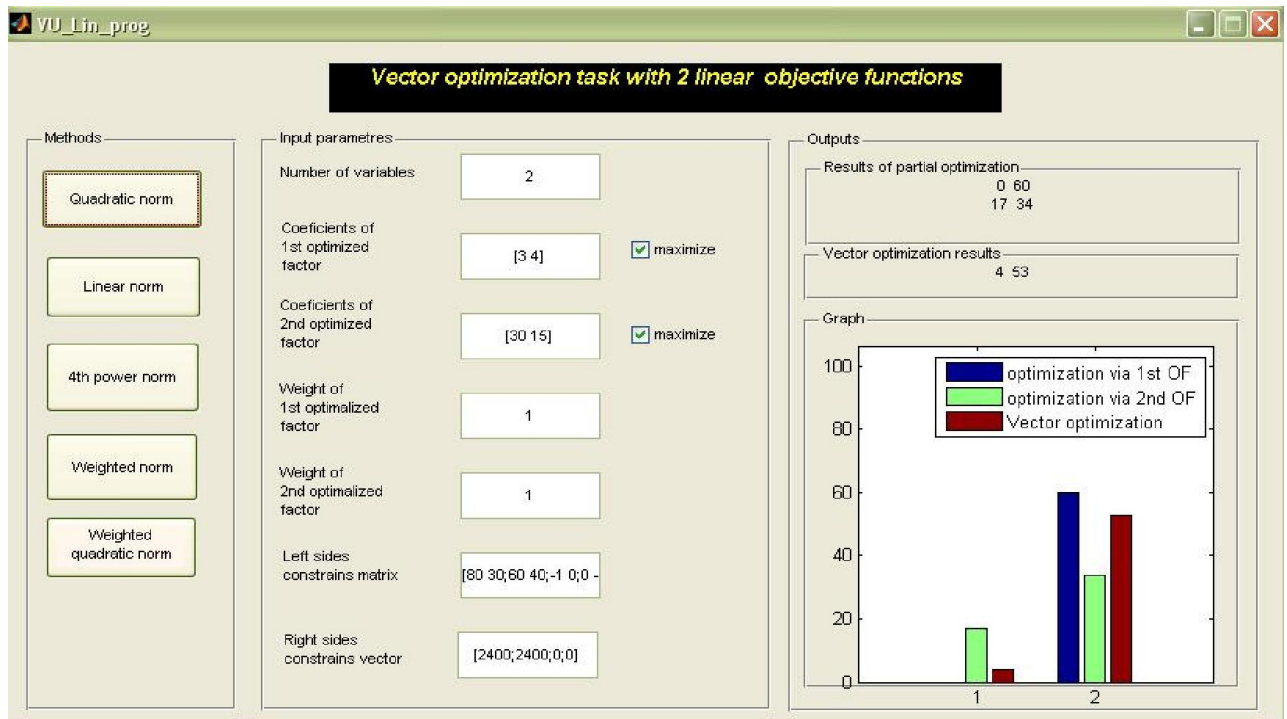

Figure 4: Solution of vector optimization task with VOPTIM application

It is also possible for user to compare results of various vector optimization methods. Results of this comparison are summed up in Table 2.

Table 2: COMPARISON OF VARIOUS VECTOR OPTIMIZATION METHODS

USED METHOD	NUMBER OF HAMMERS	NUMBER OF SCREWDRIVERS
Quadratic norm	4	53
Linear norm	17	34
4th power norm	13	40
Weighted norm(0,4;0,6)	17	34
Weighted norm(0,6;0,4)	17	34
Weighted quadratic norm(0,4;0,6)	17	34
Weighted quadratic norm(0,6;0,4)	0	60

As it is obvious from Table 2, vector optimization results depend on used method. Choosing the method mainly depends on type of problem.

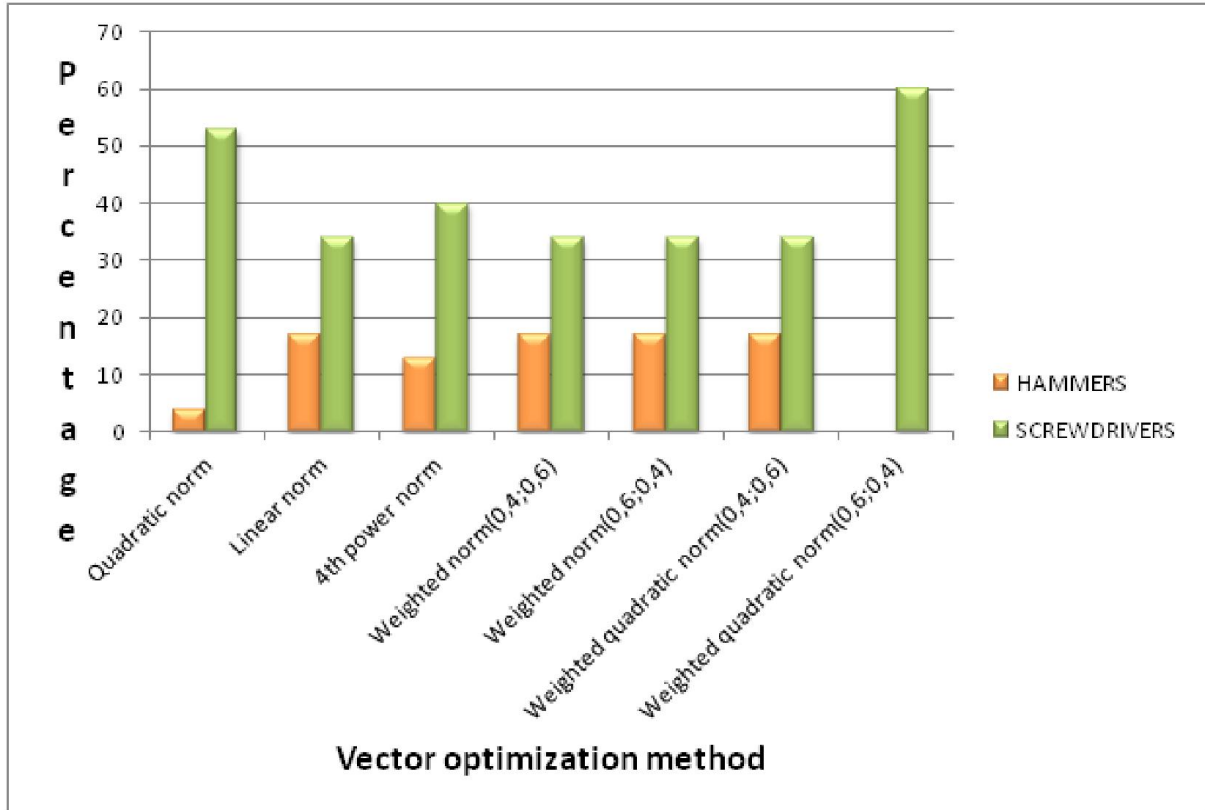Graph corresponding to the Table 2 is displayed on Figure 5.

Figure 5: Graph showing various methods of vector optimization

3.2 Ideal investment strategy problem

Investor was given some capital from the lender and he is deciding about the best investment of the funds. He can divide whole capital into three commodities. Investor has economical knowledge of Markowitz model (closely described in [8] and [9]) and his extension for safe investment. At the basis of these knowledge investor defines main conditions of his strategy:

Investor expects minimum profit of 10%. He assumes 10% profitability from investment to 1st commodity, 5% profitability from investment to 2nd commodity a 16% profitability from investment to 3rd commodity. Investor also knows variance σ_i^2 of ith commodity and covariance coefficients σ_{ij} i^{th} and j^{th} commodity. Values of these coefficients are:

$$\sigma_1^2 = 0,2; \sigma_2^2 = 0,03; \sigma_3^2 = 0,18; \ \sigma_{12} = 0,05; \ \sigma_{13} = 0,02; \ \sigma_{23} = 0,03.$$

Investor's goal is to minimize the riskiness of his investment. This can be reached by minimizing the variance of commodity portfolio. It is defined by formula

$$Var_p = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + x_3^2\sigma_3^2 + 2x_1x_2\sigma_{12} + 2x_1x_3\sigma_{13} + 2x_2x_3\sigma_{23}. \qquad (17)$$

Lender expects refunding of investment by monthly payments. He monthly claims 8% from capital invested to 1st commodity, 10% from capital invested to 2nd commodity and 7% from capital invested to 3rd commodity. Investor also doesn't want to invest more than 60% of given capital to one commodity.

What is the ideal investment strategy for investor in order to minimize the riskiness of investment and also minimize monthly payment to the lender?

Problem Solving

Solution for this problem is to find ideal values of vector x by optimizing objective functions

$$0,2\ x_1^2 + 0,08\ x_2^2 + 0,18\ x_3^2 + 0,1x_1x_2 + 0,04x_1x_3 + 0,06x_2x_3 \rightarrow \min$$

$$0,08x_1 + 0,1x_2 + 0,07x_3 \rightarrow \min$$

with respecting constrains

$$0,1x_1 + 0,05x_2 + 0,16x_3 \geq 0,1\ ;$$

$$x_1 + x_2 + x_3 = 1;$$

$$0 \leq x_1, x_{2,}x_3 \leq 0,6.$$

First step is find optimal solutions of objective functions separately, so just for

$$0,08x_1 + 0,1x_2 + 0,07x_3 \rightarrow min$$

and

$$0,2\ x_1^2 + 0,08\ x_2^2 + 0,18\ x_3^2 + 0,1x_1x_2 + 0,04x_1x_3 + 0,06x_2x_3 \rightarrow min.$$

For finding optimal solution of quadratic optimization problem is used `quadprog` MATLAB function. `Quadprog` function in MATLAB is used for solving quadratic programming tasks, generally described (in matrix notation) as:

$$\min_x \quad (\frac{1}{2}x^THx + f^Tx) \tag{18}$$

respecting constrains

$$A * x \leq b,$$

$$Aeq * x = beq,$$

$$lb \leq x \leq ub, \tag{19}$$

where H is matrix of coefficients representing quadratic term, f is vector of coefficients representing linear term, Aeq and beq are coefficients used in equalities constrains, A and b coefficients used in inequalities constrains, and lb and ub are lower and upper boundaries of searched variables.

Result of this optimization is displayed in Table 3. From the results we can claim, that in optimizing for monthly payment investor will invest most of his funds to 3rd commodity and he won't invest any of funds to 2nd one. On the other hand, in minimizing riskiness ratio, most of funds will be invested to 2nd commodity, and the smallest part will be invested to 1st one.

Table 3: SEPARATE OBJECTIVE FUNCTIONS OPTIMIZATION

	x_1	x_2	x_3
optimization for monthly payment	40%	0%	60%
optimization for riskiness ratio	19,447%	43,938%	36,615%

Another step is calculation of ideal values for both objective functions. This calculation is done by inserting ideal values counted before into objective functions.

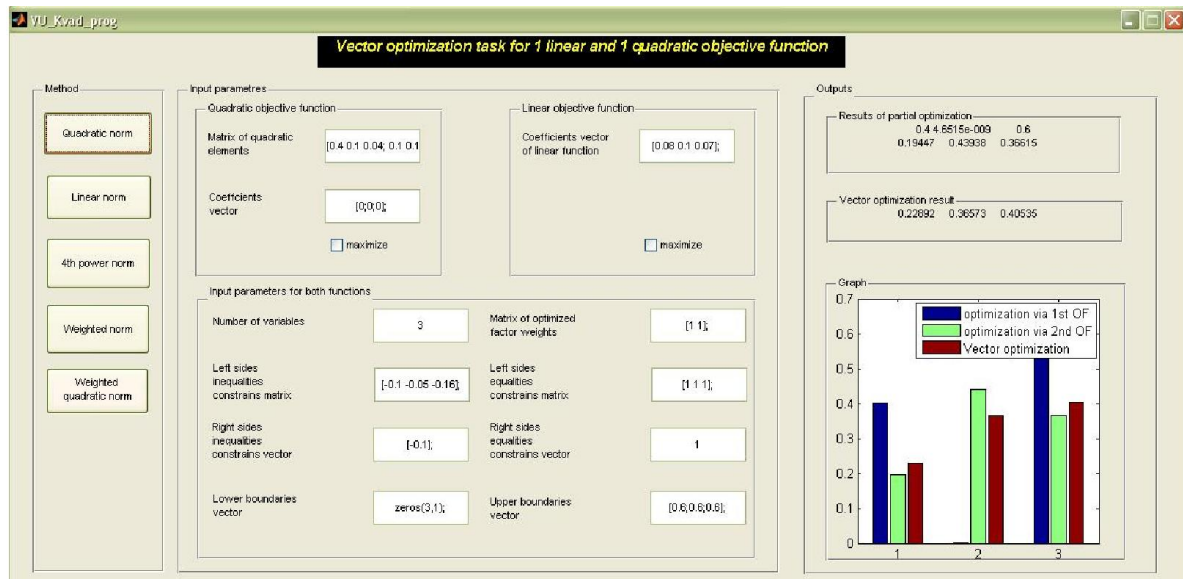User interface used to solve this vector optimization problem is displayed in Figure 6.

Figure 6: VOPTIM application interface used for solving ideal investment strategy problem

Table 4 displays comparison of vector optimization results using various vector optimization methods, specifically linear norm, 4th power norm, and weighted quadratic norm for different importance of objective functions were used (1st value of importance is for optimization of monthly payment, 2nd one is for optimization of riskiness ratio).

Table 4: IDEAL DISTRIBUTION VALUES USING OTHER NORMS OF VECTOR OPTIMIZATION

	x_1	x_2	x_3
optimization - monthly payment	40%	0%	60%
optimization - riskiness ratio	19,447%	43,938%	36,615%
VO - quadratic norm	22,892%	36,573%	40,535%
VO - linear norm	19,447%	43,938%	36,615%
VO - 4th power norm	22,373%	37,681%	39,945%
VO - weighted quadratic norm (0,4 ; 0,6)	19,447%	43,938%	36,615%
VO - weighted quadratic norm (0,6 ; 0,4)	26,206%	29,489%	44,305%
VO - weighted quadratic norm (0,7 ; 0,3)	37,315%	5,740%	56,945%

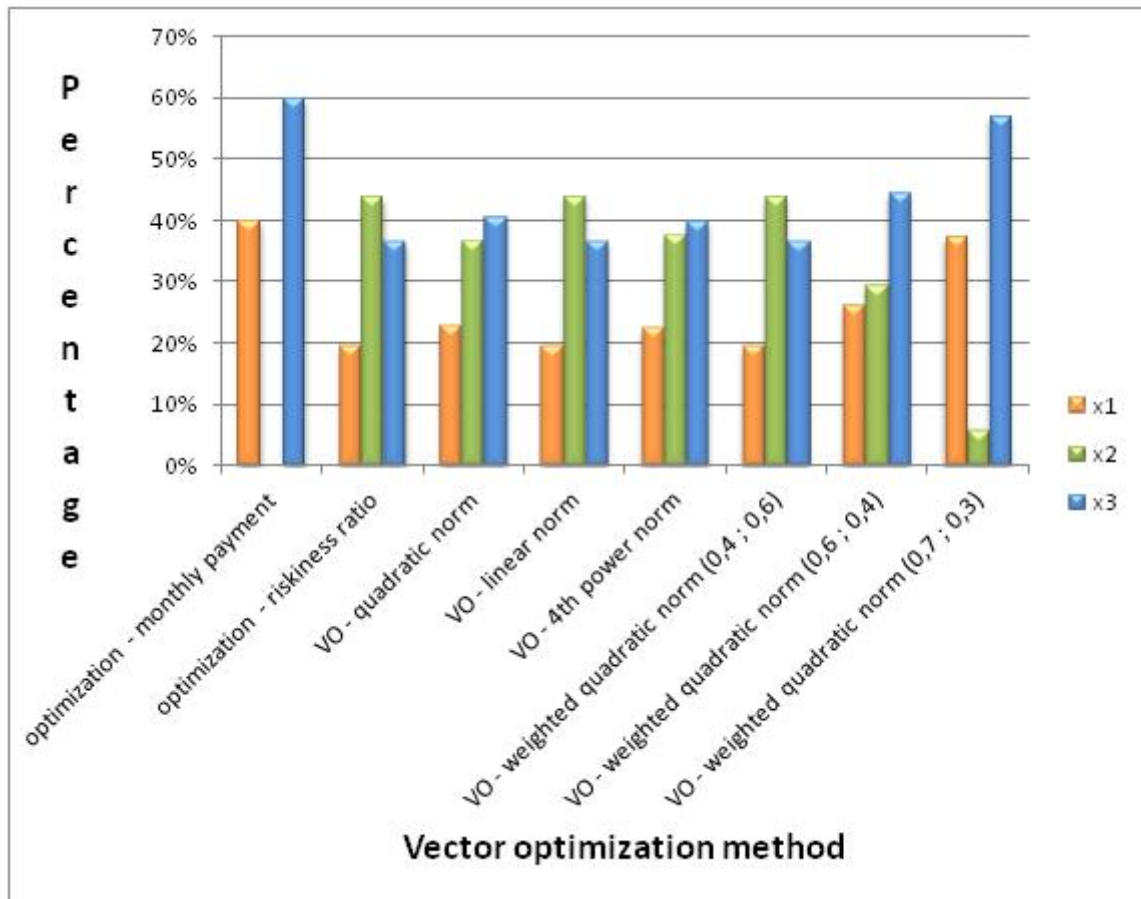Graph corresponding to the Table 4 is displayed on Figure 7.

Figure 7: Graph showing comparison of various vector optimization methods

4   Conclusion

Vector optimization is practical way to solve multiple-objective tasks. It can be used in many different fields, mainly because of its mathematical basis. Especially when there are too many objectives, which have to be considered, it is easier to delegate decision and optimization process to mathematical software. VOPTIM vector optimization tool provides users with possibility of solving vector optimization tasks with linear or quadratic objective functions with linear constrains. This fact enables user to solve various vector optimization tasks using this application.

VOPTIM application can be used in educational process in Technical University in Košice as there are many subjects dealing with optimization problems in many different application areas, from process optimization to economical optimization problems.

Acknowledgement

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